

# Aspects of Holographic Entanglement Entropy and Complexity

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Dedicated  
to  
My Parents  
and  
Teachers

## List of publications on which the thesis is based

1. Holographic information theoretic quantities for Lifshitz black hole.  
*Sourav Karar and Sunandan Gangopadhyay;*  
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2. Holographic complexity of boosted black brane and Fisher information.  
*Sourav Karar, Rohit Mishra and Sunandan Gangopadhyay;*  
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3. Holographic entanglement thermodynamics for higher dimensional charged black hole.  
*Sourav Karar, Debabrata Ghorai and Sunandan Gangopadhyay;*  
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4. Holographic complexity for Lifshitz system.  
*Sourav Karar and Sunandan Gangopadhyay;*  
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## List of publications not included in this thesis

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2. Holographic complexity of “black” non-susy D3-brane and the high temperature limit.  
*Sourav Karar, Sunandan Gangopadhyay and A.S. Majumdar;*  
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3. Entanglement thermodynamics for an excited state of Lifshitz system.  
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# Chapter 1

## Introduction and Overview

The holographic principle states that we can understand the physics inside a given volume, provided, we have a good knowledge of a particular theory at the boundary of the volume [1, 2]. One important result in black hole physics is that the entropy of a black hole is proportional to its event horizon area [3], and is independent of its volume. This result has a remarkable similarity with the holographic principle. The *AdS/CFT* correspondence, which relates the strongly correlated quantum field theory (gauge theory) to the classical gravitational theory in one higher dimension, is the first concrete realization of the holographic principle.

A conformal field theory (*CFT*) is a field theory which is invariant under conformal transformations [4]. On the other hand, the anti-de Sitter (*AdS*) spacetime is a solution of the Einstein equation with a constant negative curvature [5]. The original formulation of the *AdS/CFT* duality, proposed by Juan Maldacena, had related the  $\mathcal{N} = 4$  super conformal gauge theory in (3+1) - dimensions and the gravitational theory in five dimensional *AdS* spacetime ( $AdS_5 \times S^5$ ) [6].

The gauge theory can describe, the strong force, weak force, and the electromagnetic force. However, it is very difficult to compute a gauge theory when the theory is strongly coupled. On the other hand, in duality, a strongly coupled theory is mapped to a weakly coupled theory [7]. Therefore, the *AdS/CFT* dictionary can be used to analyze the strongly coupled gauge theory by studying the weakly coupled gravitational theory. Since its inception, it has been

used extensively to study different areas of physics, such as quantum chromodynamics, (QCD), quantum gravity, condensed matter physics, the black hole physics, relativistic fluid dynamics, information theory, etc [8]-[15].

A key point in which quantum mechanics differs from classical mechanics is the possibility of quantum entanglement. Two or more quantum particles are said to be entangled if the actions performed on any one of the particles affects the other, regardless of the distance between them. Quantum entanglement between two subsystems is characterized by a quantity called the entanglement entropy (von-Neumann entropy). Recently entanglement entropy (EE) has been an active area of research [16]-[22]. Computation of EE in  $(1 + 1)$  - dimensional gauge theory involves the replica trick [19]. However, except in a few simple cases, the computation of EE using the replica trick is difficult or impossible [23]. The famous Ryu-Takayanagi (RT) proposal gives a handle to compute the EE using the holographic principle. Consider a  $(d + 2)$  - dimensional asymptotically *AdS* spacetime ( $\mathcal{M}$ ) with a  $(d + 1)$  - dimensional conformal boundary ( $\partial\mathcal{M}$ ). Then at a constant time slice one has to chose a closed region  $A$  on  $\partial\mathcal{M}$  to define the EE between the degrees of freedom inside the region  $A$  and the rest. If  $\gamma_A$  is the minimal area at a constant time slice in  $\mathcal{M}$  with,  $\partial\gamma_A = \partial A$ , then the RT proposal states that the holographic entanglement entropy (HEE) of the subsystem  $A$  is given by [24, 14]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_{(d+2)}} , \tag{1.1}$$

where  $G_{(d+2)}$  is the  $(d + 2)$  - dimensional Newton constant. It has been argued in [25]-[28] that there is a link between the entanglement structure of quantum states in the *CFT* and the spacetime geometry of the dual bulk theory. This means that the study of EE can shed light on the unsolved mysteries in gravity.

Computational complexity is a well known term in computer science. It quantifies the difficulty of carrying out a task. In quantum mechanics, a quantum state undergoes a unitary evolution. Quantum computational complexity quantify the difficulty to implement such unitary operation. More precisely it can be interpreted as the minimum number of gates required to implement a unitary operator. In [29], the concept of computational complexity was used for the first time in black holes as an argument to counter the *AMPS* firewall proposal. The

*AMPS* firewall theory [30] is connected to the phenomena of Hawking radiation [31, 32]. The Hawking radiation talks about the formation of an entangled pair of photons near the event horizon, with one of them falling into the black hole and the other moving away from the black hole. However, after the Page time [33], the outgoing photon must be entangled with both the infalling photon and, independently, with the past Hawking radiation. This violates the principle of monogamy of entanglement, which states that any quantum system cannot be simultaneously entangled with two independent quantum systems. The *AMPS* firewall proposal was to provide a resolution of this violation, but resulted into a paradox. In [28], it was proposed that the outgoing and ingoing photons are connected by the Einstein-Rosen bridge (wormhole). Thus the photons are not independent and satisfies the principle of monogamy of entanglement.

One important property of the Einstein-Rosen bridge (ERB) is that it grows forever. On the other hand, the black hole horizon grows until it attains a final value. Then we say that the entropy has attained its maximum value and the black hole is in thermal equilibrium. This led to consider the computational complexity as the dual of the growing ERB as quantum states evolves forever and hence the complexity. The holographic computational complexity is given by [34]

$$C_V(t_L, t_R) = \frac{V(t_L, t_R)}{\ell G_N}, \quad (1.2)$$

where  $V(t_L, t_R)$  denotes the volume of ERB. This volume is the extremal volume of the codimension one bulk surface that ends on the boundary times  $t_L$  and  $t_R$ . Moreover,  $\ell$  is some length associated with the geometry of spacetime and  $G_N$  is the Newton constant. There is another notion of computational complexity, called the subregion complexity. The holographic subregion complexity (HSC) is related to the volume enclosed by the Ryu-Takayanagi extremal surface in the bulk. The HSC is given by [35]

$$C_V = \frac{V(\gamma)}{8\pi R G_N}, \quad (1.3)$$

where  $V(\gamma)$  is the volume enclosed by the Ryu-Takayanagi extremal surface,  $R$  is the radius of curvature of the spacetime and  $G_N$  is the Newton constant.

Another proposal to compute the complexity involves the computation of a classical action on the Wheeler-De Witt patch [36, 37]

$$C_W(t_L, t_R) = \frac{A_W(t_L, t_R)}{\pi \hbar} , \quad (1.4)$$

where  $A_W$  is the action evaluated on the Wheeler-De Witt patch with suitable boundary times  $(t_L, t_R)$ .

In this thesis, various aspects of holographic entanglement entropy (HEE) and holographic subregion complexity (HSC) have been studied. Various chapters in this thesis are based on the following publications [38]-[41].

The thesis has been initiated with a brief review of HEE and entanglement thermodynamics for a  $(2 + 1)$  - dimensional quantum many body system with an anisotropic scaling symmetry near its quantum critical point. A detailed computation of thermodynamical first law-like relation has been performed. The law named the first law of entanglement thermodynamics has related the change in entanglement entropy with the change in energy, entanglement pressure, and the entanglement chemical potential.

In chapter 3, which is based on the work [38], we have tried to find a relation between the change in HSC and the change in HEE. To accomplish the task we have chosen the pure Lifshitz spacetime and an asymptotic Lifshitz spacetime respectively as the ground state and excited state. A connection between the change in HSC and the change in energy, which may be regarded as an analogous relation corresponding to the first law of entanglement thermodynamics, has been found.

Holographic entanglement entropy for a  $(3 + 1)$  - dimensional Lifshitz black hole has been obtained in chapter 4. This chapter is based on the work [41]. The entanglement entropy has been studied in both the infra-red and ultra-violet limit. Near horizon behaviour of HEE has also been studied. A notion of generalized temperature has been introduced in terms of the renormalized HEE. The generalized temperature is found to be smoothly reducing to the black hole temperature in the infra-red limit.

In chapter 5, which is based on the work [39], HEE for a Reissner-Nordstrom black hole in

*AdS* spacetime has been studied. Our analysis is valid for any arbitrary dimension of spacetime. Note that in the case of the *AdS*-RN black hole, the state space of the field theory depends upon two physical parameters, namely, the charge  $Q$  and temperature  $T_H$  of the black hole. We have computed the HEE in different charge and temperature limits. At last we have obtained the first law of entanglement thermodynamics.

In chapter 6, which is based on the work [40], holographic subregion complexity for a boosted black brane has been computed. The boosted black brane has an asymmetry due to the boost parameter. This asymmetry has been investigated in the context of complexity. Holographic computation of the Fisher information metric and the fidelity susceptibility have been performed. These are important concepts of distance between two states in quantum information science. The holographic expressions for the Fisher information metric and fidelity susceptibility do not match with each other, though, they are related in general in quantum information theory. This is an important observation in this chapter.

Finally, we summarize the findings in this thesis in chapter 7.

# Chapter 2

## Review of entanglement thermodynamics for a Lifshitz system

We begin with the discussion of holographic entanglement thermodynamics for a Lifshitz system in this chapter. The discussion here is based on [42]. A system in thermal equilibrium can be described by a few macroscopic quantities, like, energy, pressure, temperature, entropy, volume, and certain chemical potentials associated with conserved charges of the system. These quantities are related to each other by certain laws called the laws of thermodynamics. First law of thermodynamics is one such law which simply describes the law of the conservation of energy. It relates the change in energy ( $dE$ ) to the change in entropy ( $dS$ ) of the system by a proportionality constant, the temperature.

On the other hand, if the system is away from thermal equilibrium the macroscopic quantities may not be well defined. However, we can still compute the entanglement entropy. For a quantum system, the quantum information can be encoded in entanglement entropy even if it is away from equilibrium [16]-[22]. The energy or the energy density of a system can also be given independently whether it is at or away from equilibrium. Therefore it is natural to ask whether there exists a thermodynamical first law like relation when the system is away from equilibrium. This question has already been addressed in affirmative in [43, 44] for some excited state of certain quantum system. They have studied the relation of the change in holographic

entanglement entropy (HEE) [24, 14] and the corresponding change in energy of the excited state of the system. It was found that the change in energy is proportional to the change in HEE for sufficiently small subsystem. The constant of proportionality was found to be related to the size of the entangling region which was identified as the inverse of entanglement temperature ( $T_{ent}$ ) to make a contact with the first law of thermodynamics.

However, the simple relation  $dE = T_{ent}dS$ , is true only for rotationally and translationally invariant excited states as studied in [43]. For anisotropic excited states, the change in entanglement entropy will contain a pressure like term in addition to the energy term, as obtained in [44, 45]. This pressure like term can be called entanglement pressure.

However, in all the cases, relativistic system was studied to find the first law of entanglement thermodynamics. In this chapter, we shall try to find a first law like relation for the excited state of a Lifshitz system in  $(3 + 1)$  - dimensions. The Lifshitz system is both non-relativistic and non-conformal. In addition, the excited state has been chosen to be non-isotropic. This leads to the addition of an extra term in the change in HEE, identified as entanglement chemical potential.

## 2.1 Lifshitz theory

In this section we will provide a brief description of the system chosen for computing HEE. One can obtain a  $(3 + 1)$  - dimensional Lifshitz metric from the bulk action ([46, 47]) of the form,

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \right) \quad (2.1)$$

where  $\Lambda$  is the cosmological constant and  $A_\mu$  is a massive gauge field. The above action can be varied to obtain the following equations of motion

$$\begin{aligned} R_{\mu\nu} &= \Lambda g_{\mu\nu} + \frac{1}{2} F_{\mu\rho} F_\nu{}^\rho - \frac{1}{8} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} + \frac{1}{2} m^2 A_\mu A_\nu \\ \nabla_\mu F^{\mu\nu} &= m^2 A^\nu \end{aligned} \quad (2.2)$$

These equations of motion have a solution of the form [48]

$$\begin{aligned}
ds^2 &= -r^{2z} dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{r^2} \\
A &= \alpha r^z dt, \quad \alpha^2 = \frac{2(z-1)}{z}
\end{aligned} \tag{2.3}$$

with a choice  $\Lambda = -\frac{1}{2}(z^2 + z + 4)$  and  $m^2 = 2z$ . It is evident from (2.3) that the solution has a scaling symmetry :  $t \rightarrow \lambda^z t$ ,  $x \rightarrow \lambda x$ ,  $y \rightarrow \lambda y$ ,  $r \rightarrow \lambda^{-1} r$ , where  $z$  is the dynamical scaling exponent and  $\alpha$  is a constant. The spacetime (2.3) is the gravity dual of a (2+1)- dimensional quantum many body system with a Lifshitz symmetry near its quantum critical point. We are considering this unperturbed Lifshitz spacetime (2.3) as the ground state.

As we are interested in obtaining a thermodynamical first law like relation, we need another spacetime which can be considered as an excited state. One may consider asymptotic Lifshitz spacetime as an excited state. We consider an asymptotic Lifshitz spacetime as prescribed in [48],

$$\begin{aligned}
ds^2 &= -r^{2z}(1 + h_{tt}(r))dt^2 + r^2(1 + h_{xx}(r))dx^2 + r^2(1 + h_{yy}(r))dy^2 + \frac{dr^2}{r^2} \\
&\quad + 2[-r^{2z}v_{1x}(r) + r^2v_{2x}(r)] dt dx + 2[-r^{2z}v_{1y}(r) + r^2v_{2y}(r)] dt dy + 2r^2h_{xy}(r) dx dy \\
A &= \alpha r^z[(1 + a_t(r) + \frac{1}{2}h_{tt}(r))dt + v_{1x}(r)dx + v_{1y}(r)dy] .
\end{aligned} \tag{2.4}$$

The asymptotically Lifshitz space-time follows when  $h_{tt}(r)$ ,  $h_{xx}(r)$ ,  $h_{yy}(r)$ ,  $v_{1x}(r)$ ,  $v_{2x}(r)$ ,  $v_{1y}(r)$ ,  $v_{2y}(r)$ ,  $h_{xy}(r)$ , and  $a_t(r)$  go to zero as  $r \rightarrow \infty$ . Here  $h_{r\mu} = 0$  gauge is chosen. Also  $a_r = 0$  has been chosen to make the perturbations constant in the boundary directions. Furthermore one can define

$$\begin{aligned}
h_{tt}(r) &= f(r) \\
h_{xx}(r) &= k(r) + t_d(r) \\
h_{yy}(r) &= k(r) - t_d(r) \\
a_t &= j(r).
\end{aligned} \tag{2.5}$$

Substituting this in eq.(2.1), one may obtain the linearized action [48]. The solutions obtained by solving the equations of motion in radial gauge, resulting from the linearized action have

different forms for  $z = 2$  and  $z \neq 2$  [48, 49, 50]. They are, for  $z = 2$ ,

$$\begin{aligned}
j(r) &= -\frac{c_1 + c_2 \ln r}{r^4}, \\
f(r) &= \frac{4c_1 - 5c_2 + 4c_2 \ln r}{12r^4}, \\
k(r) &= \frac{4c_1 + 5c_2 + 4c_2 \ln r}{24r^4}, \\
t_d(r) &= \frac{t_{d2}}{r^4}
\end{aligned} \tag{2.6}$$

and for  $z \neq 2$ ,

$$\begin{aligned}
j(r) &= -\frac{(z+1)c_1}{(z-1)r^{z+2}} - \frac{(z+1)c_2}{(z-1)r^{\frac{1}{2}(z+2+\beta_z)}}, \\
f(r) &= 4\frac{1}{(z+2)}\frac{c_1}{r^{z+2}} + 2\frac{(5z-2-\beta_z)}{(z+2+\beta_z)}\frac{c_2}{r^{\frac{1}{2}(z+2+\beta_z)}}, \\
k(r) &= 2\frac{1}{(z+2)}\frac{c_1}{r^{z+2}} - 2\frac{(3z-4-\beta_z)}{(z+2+\beta_z)}\frac{c_2}{r^{\frac{1}{2}(z+2+\beta_z)}}, \\
t_d(r) &= \frac{t_{d2}}{r^{z+2}}
\end{aligned} \tag{2.7}$$

where  $\beta_z^2 = 9z^2 - 20z + 20 = (z+2)^2 + 8(z-1)(z-2)$  and  $c_1$ ,  $c_2$  and  $t_{d2}$  are integration constants. With this setup in hand we now move on to compute the HEE.

## 2.2 Entanglement Entropy

In this section we will compute the HEE for the pure Lifshitz spacetime and the asymptotically Lifshitz spacetime. The entangling region is taken to be a straight belt of width  $l$  in the following form

$$-\frac{l}{2} \leq x \leq \frac{l}{2}, \quad 0 \leq y \leq L. \tag{2.8}$$

The prescription of computing, HEE as described in [24, 14], talks about static minimal area of the hypersurface in the bulk whose boundary coincides with the boundary of the subsystem at  $r = \infty$ . One can parametrize the extremal surface by  $x = x(r)$  as the straight belt has translational invariance along the  $y$  direction. Now we may use the metric (2.3) to write an

expression for area of the hypersurface for Lifshitz spacetime as

$$\begin{aligned}
A^{(0)} &= \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \int_0^L dy \sqrt{r'(x)^2 + r^4} \\
&= 2L \int_0^{l/2} dx r^2 \sqrt{1 + \frac{r'(x)^2}{r^4}}
\end{aligned} \tag{2.9}$$

Here  $r'(x)$  denotes derivative of  $r(x)$  with respect to  $x$ . The standard procedure of minimization determines the function  $r'(x)$  which reads

$$r'(x) = \frac{r^4}{r_t^{(0)2}} \sqrt{1 - \frac{r_t^{(0)4}}{r^4}}. \tag{2.10}$$

This can be arranged in suitable manner to have

$$x(r) = \int_{r_t^{(0)}}^r du \frac{r_t^{(0)2}}{u^4} \frac{1}{\sqrt{1 - \frac{r_t^{(0)4}}{u^4}}}, \tag{2.11}$$

where  $r_t^{(0)}$  is the turning point of the minimal hypersurface. Using the above expression for  $x(r)$  in Eq.(2.9), we get

$$\begin{aligned}
S^{(0)} &= \frac{A^{(0)}}{4G_{(4)}} \\
&= \frac{2L}{4G_{(4)}} \int_{r_t^{(0)}}^{\delta} dr \frac{\left(\frac{r}{r_t^{(0)}}\right)^2}{\sqrt{\left(\frac{r}{r_t^{(0)}}\right)^4 - 1}} \\
&= \frac{1}{4G_{(4)}} \left[ 2L\delta - \frac{5}{3} L r_t^{(0)} \right],
\end{aligned} \tag{2.12}$$

where  $\delta$  is the cutoff for the  $r$  integral. Furthermore the length of the subsystem  $l$  can be obtained from Eq. (2.10) in the following way

$$\frac{l}{2} = \int_0^{\frac{l}{2}} dx = \int_{r_t^{(0)}}^{\infty} dr \frac{1}{r^2 \sqrt{\left(\frac{r}{r_t^{(0)}}\right)^4 - 1}} = \frac{\sqrt{\pi} \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right) r_t^{(0)}}. \tag{2.13}$$

With this result in hand we now proceed to compute the HEE for the same subsystem (2.8) in asymptotic Lifshitz spacetime. The change in metric will lead to a change in HEE although we keep the same entangling subsystem. As the length  $l$  of the subsystem is fixed the change

in metric will change the turning point of the minimal hypersurface. The new turning point  $r_t$  would be  $r_t = r_t^{(0)} + \delta r_t$ , where  $\delta r_t$  is the change in the turning point.

We start by considering the minimal surface to be parametrized by  $x = x(r)$  as in the case of pure Lifshitz spacetime. The expression of area for the perturbed metric (2.4) is given by

$$A = 2L \int_0^{l/2} dx \sqrt{r'^2(x)[1 + h_{yy}(r)] + r^4[1 + h_{xx}(r) + h_{yy}(r)]}. \quad (2.14)$$

To obtain the HEE we have to minimize the above integral. This in turn determines the function  $r'(x)$  which reads

$$r'(x) = \frac{\sqrt{\frac{r^8}{Q_1}[1 + h_{xx}(r) + h_{yy}(r)]^2 - r^4[1 + h_{xx}(r) + h_{yy}(r)]}}{\sqrt{1 + h_{yy}(r)}} \quad (2.15)$$

where  $Q_1$  is a constant of integration. To fix the value of  $Q_1$  we use the fact that  $r'(x) = 0$  at the turning point  $r = r_t$ . Therefore we have

$$Q_1 = [1 + h_{xx}(r_t) + h_{yy}(r_t)]r_t^4. \quad (2.16)$$

Note that we are interested in small perturbations around the pure Lifshitz spacetime. For this the length of the strip ( $l$ ) is assumed to be small. Therefore the hypersurface in the bulk will penetrate only the ultra-violet region of the perturbed Lifshitz spacetime. This in turn means that turning point  $r_t$  lies very close to the boundary and  $h$ 's are small. Holographically this implies that we need not bother about the infra-red geometry and the asymptotic behaviour of the perturbed Lifshitz spacetime (eq.(2.4)) needs to be considered only.

Integrating eq.(2.15) from  $x = 0$  to  $x = l/2$ , we obtain upto first order in  $h$

$$\frac{l}{2} = \int_{r_t}^{\infty} \frac{dr}{r^2 f(r, r_t)} \left\{ 1 - h_{xx}(r) - \frac{1}{2}h_{yy}(r) + \frac{1}{2}[h_{xx}(r_t) + h_{yy}(r_t)] + \frac{[h_{xx}(r_t) + h_{yy}(r_t) - h_{xx}(r) - h_{yy}(r)]}{2f^2(r, r_t)} \right\} \quad (2.17)$$

where  $f^2(r, r_t) = (r/r_t)^4 - 1$ . Since we are keeping  $l$  fixed, we also have

$$\frac{l}{2} = \int_{r_t^{(0)}}^{\infty} \frac{dr}{r^2 f(r, r_t^{(0)})}. \quad (2.18)$$

Above two expressions can be equated to have

$$\delta r_t = -N r_t^{(0)} r_t \int_{r_t}^{\infty} \frac{dr}{r^2 f(r, r_t)} \left\{ h_{xx}(r) + \frac{1}{2} h_{yy}(r) - \frac{1}{2} [h_{xx}(r_t) + h_{yy}(r_t)] - \frac{[h_{xx}(r_t) + h_{yy}(r_t) - h_{xx}(r) - h_{yy}(r)]}{2f^2(r, r_t)} \right\} \quad (2.19)$$

with

$$N = \left\{ \int_1^{\infty} \frac{d\xi}{\xi^2 \sqrt{\xi^4 - 1}} \right\}^{-1} = \frac{\Gamma(1/4)}{\sqrt{\pi} \Gamma(3/4)}. \quad (2.20)$$

Setting  $\delta r_t = 0$  [44], we get the following condition on the perturbations

$$\frac{[h_{xx}(r_t) + h_{yy}(r_t)]}{2f^2(r, r_t)} = \frac{1}{[1 + f^2(r, r_t)]} \left\{ h_{xx}(r) \left( 1 + \frac{1}{2f^2(r, r_t)} \right) + \frac{h_{yy}(r)}{2} \left( 1 + \frac{1}{f^2(r, r_t)} \right) \right\} \quad (2.21)$$

Further the width of the entangling region  $l$  is same as the pure Lifshitz geometry. Since  $\delta r_t = 0$ , we have  $r_t = r_t^{(0)}$  in all the equations. Using this condition in eq.(s)(2.14, 2.15) we get area upto order  $\mathcal{O}(h)$ , given by

$$A = A^{(0)} + L \int_{r_t^{(0)}}^{\delta} dr \frac{[h_{yy}(r) + (\frac{r_t^{(0)}}{r})^4 h_{xx}(r)]}{\sqrt{1 - (\frac{r_t^{(0)}}{r})^4}} \quad (2.22)$$

and the same expression for  $x(r)$  as in eq.(2.11).

The change in HEE due to the change in background geometry is given by

$$\Delta S = \frac{A - A^{(0)}}{4G_{(4)}} = \frac{L}{4G_{(4)}} \int_{r_t^{(0)}}^{\delta} dr \frac{[h_{yy}(r) + (\frac{r_t^{(0)}}{r})^4 h_{xx}(r)]}{\sqrt{1 - (\frac{r_t^{(0)}}{r})^4}}. \quad (2.23)$$

Now we may use this expression to explicitly compute the change in HEE for  $z = 2$  and  $z \neq 2$ .

Using the expressions for  $h_{xx}(r)$  and  $h_{yy}(r)$  from (2.5)-(2.7) in (2.23) we get for  $z = 2$ ,

$$\Delta S = \frac{L}{4G_{(4)}} \frac{\sqrt{\pi} \Gamma(\frac{3}{4})}{24r_t^3 \Gamma(\frac{1}{4})} \left[ \frac{32c_1}{5} - \frac{48t_{d2}}{5} + c_2 \left( \frac{352}{25} - \frac{8\pi}{5} \right) + \frac{32}{5} c_2 \ln r_t \right] \quad (2.24)$$

and for  $z \neq 2$

$$\Delta S = \frac{L}{4G_{(4)}} \frac{\sqrt{\pi}}{2r_t^{z+1}} \left[ \frac{\Gamma(\frac{1+z}{4})}{\Gamma(\frac{3+z}{4})} \frac{1}{(z+3)} (2c_1 - t_{d2}) + r_t^{\frac{1}{2}(z+2-\beta_z)} \frac{\Gamma(\frac{z+\beta_z}{8})}{\Gamma(\frac{z+4+\beta_z}{8})} \frac{2(4 + \beta_z - 3z)}{4 + z + \beta_z} c_2 \right]. \quad (2.25)$$

## 2.3 Entanglement Thermodynamics

In this section, we will try to express the change in HEE ( $\Delta S$ ) in terms of the holographic energy-momentum tensor. This would then look like the first law (of thermodynamics) like relation involving the change in HEE and the change in energy. Varying the action with respect to the boundary metric would give us the various components of the stress tensor. The expressions for the stress tensor in terms of the functions defined earlier have the form [48],

$$\begin{aligned}
 T_{tt} &= -r^{z+2} \left[ 2r\partial_r k(r) + \alpha^2 \left( zj(r) + r\partial_r j(r) + \frac{1}{2}r\partial_r f(r) \right) \right] \\
 T_{xx} &= -2r^{z+2} \left[ (z-1)j(r) - \frac{r}{2}\partial_r f(r) - \frac{r}{2}\partial_r k(r) - \frac{1}{2}(z+2)t_d(r) \right] \\
 T_{yy} &= -2r^{z+2} \left[ (z-1)j(r) - \frac{r}{2}\partial_r f(r) - \frac{r}{2}\partial_r k(r) + \frac{1}{2}(z+2)t_d(r) \right]. \quad (2.26)
 \end{aligned}$$

It should be noted that the stress tensor complex of Lifshitz theory has a contribution from the massive gauge field. The linear action may be varied with respect to the massive gauge field to obtain another function given by

$$s_0(r) = \alpha r^{z+2} \left[ zj(r) + r\partial_r \left( \frac{1}{2}f(r) + j(r) \right) \right] \quad (2.27)$$

It should be noted that the above term is not conserved though it will be useful to express the change in HEE as a first law of thermodynamics. Moreover the above expressions for the various components of the stress tensor is true only for the vacuum solution. The analysis of stress tensor when a scalar field of mass  $m$  is added to the gravity action (2.1) has been considered in [51]. It has been pointed out there that a  $z$ -dependent Breitenlohner-Freedman bound [52, 53] is required for the finiteness of the scalar field action in Lifshitz spacetime. However, a thorough analysis of the stress tensor for Lifshitz spacetime coupled with a scalar field has been missing in the literature. Hence what happens if matter is coupled to the Lifshitz geometry would require an in depth analysis and therefore would warrant a separate investigation by itself. One should keep in mind that our results are indeed valid for the vacuum solution.

Now we rewrite the expression for  $\Delta S$  (see (2.24)) by using (2.13) to replace one  $r_t$  in the

denominator. The expression for  $\Delta S$  for  $z = 2$  takes the form

$$\Delta S = \frac{lL}{12r_t^2} \frac{1}{16G_{(4)}} \left[ \frac{32c_1}{5} - \frac{48t_{d2}}{5} + c_2 \left( \frac{352}{25} - \frac{8\pi}{5} \right) + \frac{32}{5} c_2 \ln r_t \right] \quad (2.28)$$

To express the change in HEE in terms of the holographic stress tensors, we substitute the functions given in (2.6) for  $z = 2$  in (2.26) to obtain [48],

$$\begin{aligned} \langle T_{tt} \rangle &= \frac{1}{16\pi G_{(4)}} \frac{4c_2}{3} \\ \langle T_{xx} \rangle &= \frac{1}{16\pi G_{(4)}} \left( \frac{4c_2}{3} + 4t_{d2} \right). \end{aligned} \quad (2.29)$$

Using these in (2.28) we get,

$$\Delta S = \frac{\pi}{12r_t^2} \frac{(324 - 30\pi)}{25} \left[ \langle T_{tt} \rangle Ll - \frac{10}{(54 - 5\pi)} \langle T_{xx} \rangle Ll + \frac{80}{(162 - 15\pi)} \frac{Ll}{16\pi G_{(4)}} (c_1 + c_2 \ln r_t) \right] \quad (2.30)$$

The total energy and entanglement pressure are defined by

$$\begin{aligned} \Delta E &= \int_0^L \int_{-l/2}^{l/2} dy dx \langle T_{tt} \rangle \\ &= Ll \langle T_{tt} \rangle \\ \Delta P_x &= \langle T_{xx} \rangle. \end{aligned} \quad (2.31)$$

Furthermore the entanglement temperature can be defined using (2.13) as [43, 44],

$$T_{\text{ent}} = \frac{12r_t^2}{\pi} \frac{25}{(324 - 30\pi)} = \frac{48\Gamma^2\left(\frac{3}{4}\right)}{l^2\Gamma^2\left(\frac{1}{4}\right)} \frac{25}{(324 - 30\pi)} \quad (2.32)$$

to rewrite Eq.(2.30) in the following way

$$\Delta E = T_{\text{ent}} \Delta S + \frac{10}{(54 - 5\pi)} \Delta P_x V - \frac{160}{(324 - 30\pi)} \frac{Ll}{16\pi G_{(4)}} (c_1 + c_2 \ln r_t), \quad (2.33)$$

where  $V = Ll$  is the volume of the entangling region at the boundary. This looks similar to the first law of entanglement thermodynamics with an extra term. It is interesting to note that  $s_0(r)$  for  $z = 2$  is given by

$$s_0(r) = \frac{4}{3} (c_1 + c_2 \ln r), \quad (2.34)$$

where we have used (2.6) in (2.27). The expression for  $s_0(r)$  for  $z = 2$  looks similar to the last term in Eq.(2.33) provided we evaluate  $s_0(r)$  at the turning point  $r = r_t$ . Furthermore

$s_0(r_t)/(16\pi G_{(4)})$  has the same scaling dimension as the energy density or the chemical potential.

So we may define the chemical potential of the excited state as

$$\Delta\mu \equiv \frac{s_0(r_t)}{16\pi G_{(4)}} \quad (2.35)$$

to express eq.(2.33) as

$$\Delta E = T_{\text{ent}}\Delta S + \frac{10}{(54 - 5\pi)}\Delta P_x V - \frac{10}{(108 - 10\pi)}\Delta\mu Q, \quad (2.36)$$

where we have defined  $Q = m^2\alpha V = 4V$  in the last term, with  $m$  and  $\alpha$  as expressed in eqs. (2.3) and (2.2). Also the equation of motion (2.2) can be used to identify  $m^2 A^0 = m^2\alpha$  as some the charge density  $j^0$ . Therefore we may say that eq.(2.36) represents a modified first law of entanglement thermodynamics for the Lifshitz spacetime for  $z = 2$ .

Let us now proceed to find a similar kind of relation for  $z \neq 2$ . We rewrite the expression for the change in HEE for  $z \neq 2$  by replacing one  $r_t$  in the denominator of (2.25) as

$$\Delta S = \frac{Ll\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})r_t^z} \frac{1}{16G_{(4)}} \left[ \frac{\Gamma(\frac{1+z}{4})}{\Gamma(\frac{3+z}{4})} \frac{1}{(z+3)} (2c_1 - t_{d2}) + r_t^{\frac{1}{2}(z+2-\beta_z)} \frac{\Gamma(\frac{z+\beta_z}{8})}{\Gamma(\frac{z+4+\beta_z}{8})} \frac{2(4+\beta_z-3z)}{4+z+\beta_z} c_2 \right]. \quad (2.37)$$

We have to express the change in HEE ( $\Delta S$ ) in terms of the stress tensor as before. The holographic energy momentum tensor and the chemical potential for  $z \neq 2$  can be calculated as before from (2.26) and (2.27) using the functions given in (2.7) as,

$$\begin{aligned} \langle T_{tt} \rangle &= \frac{1}{16\pi G_4} \frac{4(z-2)}{z} c_1 \\ \langle T_{xx} \rangle &= \frac{1}{16\pi G_4} [2(z-2)c_1 + (z+2)t_{d2}] \\ \Delta\mu &= \frac{1}{16\pi G_4} \langle s_0(r_t) \rangle = \frac{1}{16\pi G_4} \frac{\alpha}{z-1} \left[ 4c_1 + c_2 z (4 + \beta_z - 3z) r_t^{\frac{1}{2}(z+2-\beta_z)} \right]. \end{aligned} \quad (2.38)$$

We invert the relations (2.38) to obtain

$$\begin{aligned} c_1 &= \frac{16\pi G_4 z}{4(z-2)} \langle T_{tt} \rangle, \\ t_{d2} &= \frac{16\pi G_4 (2\langle T_{xx} \rangle - z\langle T_{tt} \rangle)}{2(z+2)}, \\ c_2 &= \frac{16\pi G_4 [(z-2)(z-1)\Delta\mu - \alpha z\langle T_{tt} \rangle]}{\alpha z(z-2)(4+\beta_z-3z)r_t^{\frac{1}{2}(z+2-\beta_z)}}. \end{aligned} \quad (2.39)$$

Now eq.(2.39) can be used to rewrite eq.(2.37) in the following way

$$\Delta S = \frac{\pi\Gamma\left(\frac{1}{4}\right)K_1}{r_t^z\Gamma\left(\frac{3}{4}\right)} \left[ \langle T_{tt} \rangle Ll - \frac{K_2}{K_1} \langle T_{xx} \rangle Ll + \frac{K_3}{K_1} \Delta\mu(m^2\alpha Ll) \right] \quad (2.40)$$

where

$$\begin{aligned} K_1 &= \frac{z^2}{(z+3)(z^2-4)} \frac{\Gamma\left(\frac{1+z}{4}\right)}{\Gamma\left(\frac{3+z}{4}\right)} - \frac{2}{(z-2)(4+z+\beta_z)} \frac{\Gamma\left(\frac{z+\beta_z}{8}\right)}{\Gamma\left(\frac{z+4+\beta_z}{8}\right)}, \\ K_2 &= \frac{1}{(z+3)(z+2)} \frac{\Gamma\left(\frac{1+z}{4}\right)}{\Gamma\left(\frac{3+z}{4}\right)}, \quad K_3 = \frac{1}{2z(4+z+\beta_z)} \frac{\Gamma\left(\frac{z+\beta_z}{8}\right)}{\Gamma\left(\frac{z+4+\beta_z}{8}\right)} \end{aligned} \quad (2.41)$$

The entanglement temperature can be defined as [43, 44],

$$T_{\text{ent}} = \frac{r_t^z\Gamma\left(\frac{3}{4}\right)}{\pi\Gamma\left(\frac{1}{4}\right)K_1} \quad (2.42)$$

to rewrite (2.40) as,

$$\Delta E = T_{\text{ent}}\Delta S + \frac{K_2}{K_1}\Delta P_x V - \frac{K_3}{K_1}\Delta\mu Q \quad (2.43)$$

where we have again used  $Q = m^2\alpha V = 2\sqrt{2z(z-1)}V$ . This may be called the modified first law of entanglement thermodynamics [44] for Lifshitz system with  $z \neq 2$ .

## 2.4 Summary

In this chapter, we have reviewed the modified first law of entanglement thermodynamics for the excited state of a Lifshitz spacetime in  $(3+1)$  - dimensions. The pure Lifshitz spacetime is considered to be the gravity dual of the ground state, and the gravity dual of the excited state is considered to be the asymptotically Lifshitz spacetime. First, holographic entanglement entropy (HEE) for a strip-like subsystem has been computed for the ground state and excited state by the famous Ryu-Takayanagi proposal. Then the change in HEE has been related to the change in energy and the change in entanglement pressure. It has been observed that the change in HEE for the Lifshitz system contains an extra term that was absent for a relativistic system. The extra term has been identified as the change in entanglement chemical potential times a charge  $Q$ .

## Chapter 3

# Holographic subregion complexity for Lifshitz system

In the previous chapter, we have discussed the modified form of the first law of entanglement thermodynamics for an asymptotic Lifshitz spacetime. The law relates the change in holographic entanglement entropy (HEE) to the change in energy, entanglement pressure and entanglement chemical potential. It is well known that quantum entanglement and quantum complexity are fundamental quantities in information theory. Furthermore, for a long time information theory has been used to understand the fundamental laws of physics, from the thermodynamic derivation of Einstein's equations of general relativity [55] to the black hole information paradox [56]. Therefore, it is natural to ask, what happens to the holographic subregion complexity (HSC) if we change the background geometry keeping the entanglement region unchanged and whether the change in HSC can be related to the change in HEE. We think that it is an interesting study since this would reveal the relation between complexity and entanglement entropy (EE).

In this chapter (based on the work in [38]), we shall first try to find the change in HSC due to the change in background geometry and then a relation similar to the first law of entanglement thermodynamics. We choose the non-relativistic Lifshitz system in  $(3 + 1)$ - dimensions for our purpose. Such theories have a lot of importance in the study of condensed matter systems near

the quantum critical point. We shall first compute HSC for the pure Lifshitz spacetime and then for the asymptotically Lifshitz spacetime. Then we would find the change in the HSC between the perturbed and the pure Lifshitz spacetimes, given by

$$\Delta C_V = C(\gamma_{ALS}) - C(\gamma_{LS}) \quad (3.1)$$

where  $\gamma_{ALS}$  is the minimal hypersurface for the asymptotic Lifshitz spacetime and  $\gamma_{LS}$  is the minimal hypersurface for the pure Lifshitz geometry. As both  $C(\gamma_{LS})$  and  $C(\gamma_{ALS})$  are divergent quantities, the difference in HSC  $\Delta C_V$  is physically more relevant, since, their difference is a finite quantity. We would then try to relate this change in HSC with changes in the HEE, energy, and the entanglement chemical potential. A similar kind of investigation has been carried out in [57] for the near horizon geometry of  $D3$ -brane shell. Further, Lifshitz spacetime has a dynamic scaling exponent  $z$ . We shall carry out the analysis for both  $z = 2$  and  $z \neq 2$ . The chapter is organized as follows. In the next section, we compute the change in the HSC between an excited state of the Lifshitz spacetime and the ground state. We then relate it to the components of the holographic stress tensor. We summarize in section (3.3).

### 3.1 Holographic subregion complexity for Lifshitz spacetime

In this section we will first compute the HSC for the pure Lifshitz spacetime in  $(3+1)$ - dimensions (see eq.(2.3)) and then for the asymptotic Lifshitz spacetime (see eq.(2.4)). A strip like entangling region has been considered for both these computations. Here we have considered the pure Lifshitz spacetime to be the ground state and the asymptotic Lifshitz spacetime to be the excited state. Further, we have considered the entangling region to be unchanged in both the cases. This means that we are interested in the change in HSC due to the background geometry only.

The entangling region in the boundary is taken to be a straight belt with width  $l$  such that  $-\frac{l}{2} \leq x \leq \frac{l}{2}$  and  $0 \leq y \leq L$ , which is identical with the choice in (2.8). As in the previous

chapter, we have parametrized the profile of the extremal surface by  $x = x(r)$ . With this set up in hand, we now proceed to compute the HSC for pure Lifshitz spacetime, which is given by

$$C_V^{(0)} = \frac{V^{(0)}}{8\pi G_{(4)}}; \quad V^{(0)} = 2L \int_{r_t^{(0)}}^{\infty} dr r x(r) \quad (3.2)$$

where  $r_t^{(0)}$  is the turning point at which  $r'(x) = 0$ . The above integral involves the profile of the extremal surface  $x(r)$ . We already know that computing the HSC means computation of the volume enclosed by the minimal hypersurface whose boundary coincides with the boundary of the subsystem lying at the boundary at a constant time slice. Therefore, the extremal surface is the same as the extremal surface involving the computation of HEE for the pure Lifshitz spacetime (see eq. (2.11)), which is given by

$$x(r) = \int_{r_t^{(0)}}^r du \frac{r_t^{(0)2}}{u^4} \frac{1}{\sqrt{1 - \frac{r_t^{(0)4}}{u^4}}} . \quad (3.3)$$

Substituting the above expression for  $x(r)$  in eq.(3.2) , we have

$$\begin{aligned} V^{(0)} &= 2L \int_{r_t^{(0)}}^{\delta} dr r \int_{r_t^{(0)}}^r du \frac{r_t^{(0)2}}{u^4} \frac{1}{\sqrt{1 - \frac{r_t^{(0)4}}{u^4}}} \\ &= \frac{\sqrt{\pi} \Gamma\left(\frac{3}{4}\right) L \delta^2}{\Gamma\left(\frac{1}{4}\right) r_t^{(0)}} - \frac{\sqrt{\pi} \Gamma\left(\frac{5}{4}\right) L r_t^{(0)}}{\Gamma\left(\frac{3}{4}\right)} \end{aligned} \quad (3.4)$$

where cut-off  $\delta$  has been introduced for the  $r$  integral and we have ignored terms of order  $1/\delta$  since they are small. From eq.(3.3), we also have

$$\frac{l}{2} = \int_0^{\frac{l}{2}} dx = \int_{r_t^{(0)}}^{\infty} dr \frac{1}{r^2 \sqrt{\left(\frac{r}{r_t^{(0)}}\right)^4 - 1}} = \frac{\sqrt{\pi} \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right) r_t^{(0)}} . \quad (3.5)$$

Therefore, the HSC for pure Lifshitz spacetime can be obtained by substituting eq.(3.4) in eq.(3.2), which is as follows

$$\begin{aligned} C_V^{(0)} &= \frac{V^{(0)}}{8\pi G_{(4)}} \\ &= \frac{\sqrt{\pi}}{8\pi G_{(4)}} \left[ \frac{\Gamma\left(\frac{3}{4}\right) L \delta^2}{\Gamma\left(\frac{1}{4}\right) r_t^{(0)}} - \frac{\Gamma\left(\frac{5}{4}\right) L r_t^{(0)}}{\Gamma\left(\frac{3}{4}\right)} \right] . \end{aligned} \quad (3.6)$$

With this result in hand we shall now proceed to calculate the HSC for the asymptotic Lifshitz spacetime given by [48]

$$\begin{aligned}
ds^2 &= -r^{2z}(1 + h_{tt}(r))dt^2 + r^2(1 + h_{xx}(r))dx^2 + r^2(1 + h_{yy}(r))dy^2 + \frac{dr^2}{r^2} \\
&\quad + 2[-r^{2z}v_{1x}(r) + r^2v_{2x}(r)] dt dx + 2[-r^{2z}v_{1y}(r) + r^2v_{2y}(r)] dt dy + 2r^2h_{xy}(r) dx dy \\
A &= \alpha r^z[(1 + a_t(r) + \frac{1}{2}h_{tt}(r))dt + v_{1x}(r)dx + v_{1y}(r)dy] .
\end{aligned} \tag{3.7}$$

This is the same metric we have used in the previous chapter (see eq.(2.4)). It is evident from eq.(3.7) that the asymptotically Lifshitz space-time follows when  $h_{tt}(r)$ ,  $h_{xx}(r)$ ,  $h_{yy}(r)$ ,  $v_{1x}(r)$ ,  $v_{2x}(r)$ ,  $v_{1y}(r)$ ,  $v_{2y}(r)$ ,  $h_{xy}(r)$ , and  $a_t(r)$  go to zero as  $r \rightarrow \infty$ . This asymptotic Lifshitz metric can be considered as the excited state of the Lifshitz spacetime. This in turn would lead to the change in complexity due to the change in the metric.

We shall consider the minimal surface to be parametrized by  $x = x(r)$  as we did in the case of pure Lifshitz spacetime. Therefore, the volume enclosed by the RT surface for the metric (3.7) is given by

$$V = \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \int_0^L dy \int_{r_t}^{\delta} dr r x(r) \sqrt{1 + h_{xx}(r) + h_{yy}(r)} . \tag{3.8}$$

We have already made clear that we are interested in the change in HSC due to the change in background only. Hence we shall keep the length of the entangling region  $l$  fixed because HSC depends on the turning point  $r_t^{(0)}$  (see eq.(3.4)) of the extremal hypersurface which in turn depends on the length of the entangling region  $l$  (see eq.(3.5)). Therefore, the turning point will change as soon as the metric is perturbed keeping the entangling region unchanged. If we consider  $r_t$  to be the new turning point, then  $r_t = r_t^{(0)} + \delta r_t$ , where  $\delta r_t$  is the change in the turning point. The expression for  $\delta r_t$  has already been obtained in the previous chapter (see eq. (2.19)), which is as follows

$$\begin{aligned}
\delta r_t &= -N r_t^{(0)} r_t \int_{r_t}^{\infty} \frac{dr}{r^2 f(r, r_t)} \left\{ h_{xx}(r) + \frac{1}{2} h_{yy}(r) - \frac{1}{2} [h_{xx}(r_t) + h_{yy}(r_t)] \right. \\
&\quad \left. - \frac{[h_{xx}(r_t) + h_{yy}(r_t) - h_{xx}(r) - h_{yy}(r)]}{2f^2(r, r_t)} \right\}
\end{aligned} \tag{3.9}$$

with

$$N = \left\{ \int_1^{\infty} \frac{d\xi}{\xi^2 \sqrt{\xi^4 - 1}} \right\}^{-1} = \frac{\Gamma(1/4)}{\sqrt{\pi} \Gamma(3/4)} . \tag{3.10}$$

Now fixing  $l$  requires  $\delta r_t = 0$  [44]. This criteria gives us a condition on the perturbation parameters:

$$\frac{[h_{xx}(r_t) + h_{yy}(r_t)]}{2f^2(r, r_t)} = \frac{1}{[1 + f^2(r, r_t)]} \left\{ h_{xx}(r) \left( 1 + \frac{1}{2f^2(r, r_t)} \right) + \frac{h_{yy}(r)}{2} \left( 1 + \frac{1}{f^2(r, r_t)} \right) \right\} \quad (3.11)$$

where  $f^2(r, r_t) = (r/r_t)^4 - 1$ . Moreover the expression for  $x(r)$  is same as eq.(3.3). We substitute eq.(3.3) in eq.(3.8) and using eq.(3.11), we obtain

$$V = V^{(0)} + L \int_{r_t^{(0)}}^{\delta} dr r [h_{xx}(r) + h_{yy}(r)] \int_{r_t^{(0)}}^r du \frac{r_t^{(0)2}}{u^4} \frac{1}{\sqrt{1 - \frac{r_t^{(0)4}}{u^4}}}, \quad (3.12)$$

where terms upto first order in  $h$  has been kept. Note that we are taking our subsystem length  $l$  to be small. Since the turning point  $r_t$  is inversely related to  $l$ , it implies that  $r_t$  lies close to the boundary ( $r = \infty$ ). This means that the perturbation parameters  $h(r)$ 's are small enough to keep only first order terms in the expression for volume.

Therefore, the change in HSC due to the change in metric is given by

$$\begin{aligned} \Delta C_V &= \frac{V - V^{(0)}}{8\pi G_{(4)}} \\ &= \frac{L}{8\pi G_{(4)}} \int_{r_t^{(0)}}^{\delta} dr r [h_{xx}(r) + h_{yy}(r)] \int_{r_t^{(0)}}^r du \frac{r_t^{(0)2}}{u^4} \frac{1}{\sqrt{1 - \frac{r_t^{(0)4}}{u^4}}}. \end{aligned} \quad (3.13)$$

We may now use the expressions for  $h_{xx}(r)$  and  $h_{yy}(r)$  from eq.(s)((2.5),(2.6),(2.7)) to explicitly write the expression for the change in HSC, which is for  $z = 2$

$$\Delta C_V = \frac{\sqrt{\pi} \Gamma(\frac{5}{4}) L}{8\pi G_4 \Gamma(\frac{7}{4}) r_t^{(0)3}} \left( \frac{c_1 + c_2 \ln r_t^{(0)}}{24} + \frac{3\pi + 13}{288} c_2 \right) \quad (3.14)$$

and for  $z \neq 2$

$$\Delta C_V = \frac{\sqrt{\pi} L}{8\pi G_4 r_t^{(0)z+1}} \left[ \frac{c_1 \Gamma(\frac{z+3}{4})}{z(z+2)\Gamma(\frac{z+5}{4})} + \frac{2c_2(4 + \beta_z - 3z)\Gamma(\frac{z+\beta_z+4}{8})}{(z + \beta_z + 2)(z + \beta_z - 2)\Gamma(\frac{z+\beta_z+8}{8})} \right]. \quad (3.15)$$

A few observations are in order at this point. Although  $V$  and  $V^{(0)}$  (eq.(s)(3.4, 3.12)) are divergent quantities, the change in HSC  $\Delta C_V$  is finite. This finiteness of the change in HSC is crucial for proposing a holographic dual of the Fisher information metric as pointed out in

[58]. In quantum information literature the Fisher information metric is an important notion of distance between two quantum systems. The Fisher information metric is given by [59]-[60],

$$G_{F,\lambda_i\lambda_j}(\lambda) = \int_x p(x, \lambda) \frac{\partial}{\partial \lambda_i} \log p(x, \lambda) \frac{\partial}{\partial \lambda_j} \log p(x, \lambda) dx \quad (3.16)$$

where  $p(x, \lambda)$  is the pure state density matrix corresponding to the pure state  $|\psi_{\lambda_i}(x)\rangle$  of a quantum mechanical system characterized by the parameters  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $n$  denotes the dimension of the statistical manifold. The proposal for the holographic computation of the Fisher information metric reads [58],

$$G_{F,mm}(\lambda) = \partial_m^2 \mathcal{F} \quad (3.17)$$

where  $m$  is the perturbation parameter in the bulk geometry and  $\mathcal{F} \sim \Delta C_V$  (upto prefactors). Moreover, similar kind of relation for the change in HEE has already been observed in the previous chapter (see eq. (2.24) and (2.25)). However, the expressions for change in HEE contained a term involving  $t_{d2}$  which is absent in the above expressions for the change in HSC.

In the next section, we shall find that the expressions for the change in HSC can be related to the change in HEE. Further, we shall also see that the change in HSC can be related to the change in energy and entanglement chemical potential. Therefore the relation can be regarded as an analogous expression corresponding to the first law of entanglement thermodynamics.

## 3.2 Change in holographic subregion complexity

We shall now express the change in HSC ( $\Delta C_V$ ) in terms of the holographic stress tensor. In the previous chapter, we have presented the various components of the stress tensor for the Lifshitz spacetime. It is known that the stress tensor complex for Lifshitz spacetime has a contribution from both, the metric (see eq.(2.26)), and the massive gauge field (see eq.(2.27)). Moreover, they are valid only for the vacuum solution.

For simplicity we rewrite the expressions for energy momentum tensor for  $z = 2$  as obtained

in the previous chapter,

$$\langle T_{xx} \rangle = \frac{1}{16\pi G_4} \left( \frac{4c_2}{3} + 4t_{d2} \right) \quad (3.18)$$

$$\langle T_{tt} \rangle = \frac{1}{16\pi G_4} \frac{4c_2}{3} . \quad (3.19)$$

From eq.(3.19), we substitute  $c_2$  in eq.(3.14) to have

$$\Delta C_V = \frac{\sqrt{\pi} \Gamma(\frac{5}{4}) L}{192 \Gamma(\frac{7}{4}) r_t^{(0)3}} \left( \frac{c_1 + c_2 \ln r_t^{(0)}}{\pi G_4} + (3\pi + 13) \langle T_{tt} \rangle \right) . \quad (3.20)$$

Now for  $z = 2$ , the total energy, entanglement pressure, entanglement chemical potential of the excited state and charge are given by (see eqs.(2.31) and (2.35))

$$\Delta E = \int_0^L dy \int_{-l/2}^{l/2} dx \langle T_{tt} \rangle = Ll \langle T_{tt} \rangle \quad (3.21)$$

$$\Delta P_x = \langle T_{xx} \rangle \quad (3.22)$$

$$\Delta \mu = \frac{1}{12\pi G_4} \left( c_1 + c_2 \ln r_t^{(0)} \right) \quad (3.23)$$

$$Q = m^2 \alpha Ll = 4Ll . \quad (3.24)$$

Using the above identities, we can recast eq.(3.20) as

$$\Delta C_V = \frac{1}{T_{ent}} [B_1 \Delta \mu Q + B_2 \Delta E] \quad (3.25)$$

with

$$B_1 = \frac{25 \Gamma(\frac{5}{4}) \Gamma(\frac{1}{4})}{32\pi (54 - 5\pi) \Gamma(\frac{7}{4}) \Gamma(\frac{3}{4})} , \quad B_2 = \left( \frac{3\pi + 13}{3} \right) B_1 . \quad (3.26)$$

Note that we have used the same expression for entanglement temperature ( $T_{ent}$ ) as obtained during the derivation of first law of entanglement thermodynamics for an excited state of the Lifshitz spacetime, given by

$$T_{ent} = \frac{12r_t^2}{\pi} \frac{25}{(324 - 30\pi)} = \frac{48\Gamma^2(\frac{3}{4})}{l^2\Gamma^2(\frac{1}{4})} \frac{25}{(324 - 30\pi)} . \quad (3.27)$$

We observe from eq.(3.25) that the change in the HSC is related to the change in energy and the change in entanglement chemical potential. We also observe that the change in pressure  $\Delta P_x$  does not appear in the expression for the change in HSC. This is in contrast to the expression

for the change in the HEE (see eq.(2.36)). Note that the presence of  $t_{d2}$  term (related to  $\Delta P_x$ ) in  $\Delta S$  is responsible for the origin of  $\Delta P_x$  in the change in HEE ( $\Delta S$ ). The  $t_{d2}$  term arise due to the presence of combination  $h_{yy}(r) + \left(\frac{r_t}{r}\right)^4 h_{xx}(r)$  (see eq.(2.23)) in  $\Delta S$ . However, in the case of  $\Delta C_V$ , the combination  $h_{xx}(r) + h_{yy}(r)$  (see eq.(3.13)) arises. This combination leads to the cancellation of the  $t_{d2}$  term responsible for the origin of  $\Delta P_x$ .

Recall the first law of entanglement thermodynamics for Lifshitz spacetime for  $z = 2$  which is given by [42]

$$\Delta E = T_{\text{ent}}\Delta S + \frac{10}{(54 - 5\pi)}\Delta P_x V - \frac{10}{(108 - 10\pi)}\Delta\mu Q . \quad (3.28)$$

We can recast eq.(3.25) using the above equation in the following form

$$\Delta C_V = B_1\Delta S + \frac{10 B_1}{54 - 5\pi} \frac{\Delta P_x V}{T_{\text{ent}}} + \left( B_2 - \frac{5B_1}{54 - 5\pi} \right) \frac{\Delta\mu Q}{T_{\text{ent}}} . \quad (3.29)$$

This expression relates the change in the HSC with the change in HEE, change in entanglement pressure and the change in entanglement chemical potential.

We can now proceed to obtain similar form of relation for the case with  $z \neq 2$ . The expressions for energy-momentum tensor and the entanglement chemical potential for  $z \neq 2$  was obtained in the previous chapter, which reads

$$\langle T_{tt} \rangle = \frac{1}{4\pi G_4} \left( \frac{z-2}{z} \right) c_1 \quad (3.30)$$

$$\Delta\mu = \frac{1}{16\pi G_4} \frac{\alpha}{[z-1]} \left[ 4c_1 + c_2 z(4 + \beta_z - 3z) r_t^{(0)\frac{1}{2}(z+2-\beta_z)} \right] . \quad (3.31)$$

Inverting the above expressions we get,

$$\begin{aligned} c_1 &= 4\pi G_4 \frac{z}{z-2} \langle T_{tt} \rangle \\ c_2 &= \frac{16\pi G_4 [(z-1)(z-2)\Delta\mu - \alpha z \langle T_{tt} \rangle]}{\alpha z(z-2)(4 + \beta_z - 3z) r_t^{\frac{1}{2}(z+2-\beta_z)}} . \end{aligned} \quad (3.32)$$

In the above expression we have represented  $c_1$  and  $c_2$  as functions of  $\langle T_{tt} \rangle$  and  $\Delta\mu$ . Now the first law of entanglement thermodynamics for  $z \neq 2$  is given by [42]

$$\Delta E = T_{\text{ent}}\Delta S + \frac{K_2}{K_1}\Delta P_x V - \frac{K_3}{K_1}\Delta\mu Q \quad (3.33)$$

where

$$\begin{aligned}
K_1 &= \frac{z^2}{(z+3)(z^2-4)} \frac{\Gamma\left(\frac{1+z}{4}\right)}{\Gamma\left(\frac{3+z}{4}\right)} - \frac{2}{(z-2)(4+z+\beta_z)} \frac{\Gamma\left(\frac{z+\beta_z}{8}\right)}{\Gamma\left(\frac{z+4+\beta_z}{8}\right)}, \\
K_2 &= \frac{1}{(z+3)(z+2)} \frac{\Gamma\left(\frac{1+z}{4}\right)}{\Gamma\left(\frac{3+z}{4}\right)}, \quad K_3 = \frac{1}{2z(4+z+\beta_z)} \frac{\Gamma\left(\frac{z+\beta_z}{8}\right)}{\Gamma\left(\frac{z+4+\beta_z}{8}\right)},
\end{aligned} \tag{3.34}$$

with the entanglement temperature

$$T_{\text{ent}} = \frac{r_t^z \Gamma\left(\frac{3}{4}\right)}{\pi \Gamma\left(\frac{1}{4}\right) K_1}. \tag{3.35}$$

Using the above expression for entanglement temperature and  $c_1, c_2$  from eq.(3.32), we can write eq.(3.15) as

$$\Delta C_V = \frac{1}{T_{\text{ent}}} [D_1 \Delta E + D_2 \Delta \mu Q] \tag{3.36}$$

where  $D_1$  and  $D_2$  are given by

$$\begin{aligned}
D_1 &= \frac{2}{\pi K_1} \left[ \frac{\Gamma\left(\frac{z+3}{4}\right)}{4(z^2-4)\Gamma\left(\frac{z+5}{4}\right)} - \frac{2}{(z-2)(z+\beta_z+2)(z+\beta_z-2)} \frac{\Gamma\left(\frac{z+\beta_z+4}{8}\right)}{\Gamma\left(\frac{z+\beta_z+8}{8}\right)} \right] \\
D_2 &= \frac{1}{\pi z(z+\beta_z+2)(z+\beta_z-2)K_1} \frac{\Gamma\left(\frac{z+\beta_z+4}{8}\right)}{\Gamma\left(\frac{z+\beta_z+8}{8}\right)}
\end{aligned} \tag{3.37}$$

and we have used the fact that  $Q = m^2 \alpha L l = 2\sqrt{2z(z-1)} L l$  is the total charge. Therefore we can see that as in the case of  $z = 2$ , the change in HSC for  $z \neq 2$  is also related to the change in energy and the change in entanglement chemical potential.

Finally we can use the eqs.(3.33) and (3.36) to relate the change in HSC with the change in HEE, which is as follows

$$\Delta C_V = D_1 \Delta S + \frac{K_2 D_1}{K_1} \frac{\Delta P_x V}{T_{\text{ent}}} + \left( D_2 - \frac{K_3 D_1}{K_1} \right) \frac{\Delta \mu Q}{T_{\text{ent}}}. \tag{3.38}$$

The above expression once again relates the change in the HSC with the change in HEE, change in entanglement chemical potential and the change in entanglement pressure for  $z \neq 2$ . The relation is the counterpart of the corresponding relation (3.29) for  $z = 2$ .

### 3.3 Summary

In this chapter, we have computed the holographic subregion complexity for a  $(3 + 1)$ - dimensional Lifshitz spacetime with a strip like entanglement region. This spacetime has been considered as the ground state. Again holographic subregion complexity has been computed for an asymptotic Lifshitz spacetime for the same entangling region. The asymptotic Lifshitz spacetime has been considered as the excited state. We would like to mention that the asymptotic Lifshitz spacetime, which we have considered for our computations, has constant perturbation (along the boundary directions) to the background geometry. Then we have computed the change in complexity between the excited state and the ground state. We have computed the change in complexity up to the first order in the perturbation parameters  $h_{xx}(r)$  and  $h_{yy}(r)$ . It should be noted that the change in complexity takes place due to the change in spacetime metric, change in the turning point ( $r_t$ ) of the hypersurface (Ryu-Takayanagi surface), or due to the change in the solution of the hypersurface. However, at the leading order, the change in holographic subregion complexity is solely caused by the change in the spacetime geometry. It was also observed that though the holographic subregion complexity for both pure and excited states were divergent, their difference was finite. This finiteness of the change in complexity is essential for the prescription of the holographic dual of the Fisher information metric. The change in subregion complexity was related to the change in the energy and the entanglement chemical potential for both  $z = 2$  and  $z \neq 2$ . This may be regarded as an analogous relation corresponding to the first law of entanglement thermodynamics. Surprisingly the change in subregion complexity does not depend on the change in entanglement pressure. This is in sharp contrast with the change in holographic entanglement entropy. Moreover, the change in subregion complexity was related to the change in holographic entanglement entropy.

In our computations, we have chosen the entangling region to be a straight belt but the results would also have a similar form for a spherical disk like entangling region. This will happen since the change in geometry of the entangling region will only change the coefficients of various terms in the expressions.

# Chapter 4

## Holographic entanglement entropy, thermal entropy, generalized temperature

It is well understood that holographic entanglement entropy (HEE) obeys a first law of thermodynamics like relation in the ultra-violet (UV) limit. This has been studied extensively in [43, 44], where they have identified the relation as the first law of entanglement thermodynamics. In the course of obtaining such a relation, they have defined a new quantity called entanglement temperature ( $T_{ent}$ ). It is found that the entanglement temperature by definition depends on the subsystem size. However, the genuine thermodynamic temperature is independent of subsystem size. Therefore the entanglement temperature arising in the UV limit is not the genuine thermodynamic temperature. This has led to the investigation of certain quantities which can be connected to the real temperature. It has led to the identification of a new quantity, the generalized temperature, that agrees with the entanglement temperature in the UV limit and the Hawking temperature of the black hole in the infra-red (IR) limit [61, 62].

Most of the studies involving holographic entanglement thermodynamics are focused on relativistic spacetimes. Relatively few investigations have been done for non-relativistic systems. In [42], thermodynamics like law was obtained for a  $(3 + 1)$ - dimensional asymptotic Lifshitz

spacetime. The Lifshitz spacetime is a well-known non-relativistic spacetime. In [38], the holographic subregion complexity for the same spacetime was computed, and a relation analogous to entanglement thermodynamics was obtained in the context of holographic subregion complexity. Investigations involving the generalized temperature in [61, 62] have been performed for relativistic systems. In this chapter, we want to analyze the generalized temperature in a non-relativistic framework. The chapter is based on the work [41].

A Lifshitz black hole in  $(3 + 1)$ -dimensions with the dynamical exponent  $z = 2$  has been considered for our purpose [63]. It is to be noted that we have two reasons to consider such spacetime. First, scale invariant theories, which are not Lorentz invariant, play an important role in analyzing condensed matter systems near their critical points [64]. A Lifshitz theory exhibits such a scale invariance. Second, the analytic solution of Lifshitz black hole in  $(3 + 1)$  - dimensions with  $z = 2$  as presented in [63] is remarkably simple compared to the other analytic solutions of Lifshitz black hole ([47],[65]-[68]). We start our analysis by computing the finite part of the holographic entanglement entropy of the Lifshitz black hole. Then we have studied its infrared (IR) and ultra-violet (UV) approximation. Next, to understand the divergence structure we have investigated the near horizon behavior of the HEE. Then we have computed the difference in HEE between a Lifshitz black hole and a pure Lifshitz spacetime. This has been identified as the renormalized HEE. Then the notion of generalized temperature has been introduced from the renormalized HEE. It is found that the generalized temperature reduces to the Hawking temperature of the Lifshitz black hole in the IR limit.

## 4.1 Lifshitz black hole

Let us start with a brief description of the background geometry. We have already said that our analysis is based on a Lifshitz black hole. The action for a  $(3 + 1)$  - dimensional Lifshitz black hole is given by [63],

$$S = \frac{1}{2} \int d^4x \sqrt{-g}(R - 2\Lambda) - \int d^4x \sqrt{-g} \left( e^{-2\Phi} \frac{F^2}{4} + \frac{m^2}{2} A^2 + (e^{-2\Phi} - 1) \right) . \quad (4.1)$$

This action enjoys a simple solution, given by

$$ds^2 = -f(r)\frac{dt^2}{r^{2z}} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2 f(r)}$$

$$A = -\frac{z^2 + z + 4}{2} ; \quad \Phi = -\frac{1}{2} \log\left(1 + \frac{r^2}{r_h^2}\right) ; \quad A = \frac{f(r)}{r^2} dt \quad (4.2)$$

with

$$f(r) = 1 - \frac{r^2}{r_h^2} \quad (4.3)$$

where  $r_h$  denotes the radius of the black hole horizon and  $z$  denotes the dynamical scaling exponent. It is evident that the above metric enjoys an anisotropic scaling as  $(x, y) \rightarrow (\lambda x, \lambda y)$  and  $t \rightarrow \lambda^z t$ . As mentioned earlier,  $z$  is called the dynamical exponent, and in the subsequent discussion we shall work with  $z = 2$ . As space and time scales in different way, this solution does not obey Lorentz invariance. Therefore, the solution is non-relativistic and plays an important role in the study of condensed matter systems near the quantum critical point. In the near boundary limit ( $r \rightarrow 0$ ), the solution (4.2) reduces to that of pure Lifshitz spacetime (zero temperature metric), given by

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2}$$

$$A = -\frac{z^2 + z + 4}{2} ; \quad A = \frac{\alpha}{r^z} dt, \quad \alpha^2 = \frac{2(z-1)}{z} . \quad (4.4)$$

Moreover the solution (4.2) in (3+1)-dimensions does not have the kinetic term for the scalar, as the scalar is strongly coupled. However, the scalar wave equation can be solved exactly.

The black hole entropy and Hawking temperature of the Lifshitz black hole are given by

$$S_h = \frac{lL}{4G_{(4)}r_h^2}; \quad T_h = \frac{1}{2\pi r_h^2} . \quad (4.5)$$

To compute the HEE for a subsystem we need to choose the subsystem. We choose a strip like subsystem which lies at the boundary of the black hole. The subsystem is of length  $l$  with the following specifications

$$-\frac{l}{2} \leq x \leq \frac{l}{2}, \quad 0 \leq y \leq L. \quad (4.6)$$

Now the HEE is proportional to the minimal area of the hypersurface in the bulk whose boundary coincides with the boundary of the subsystem at  $r = 0$  at a fixed time [24, 14]. Let the

hypersurface be parametrized by  $r = r(x)$ , leaving the  $y$ -direction independent. Using this parametrization we can write an expression for the area of the hypersurface, given by

$$A = L \int_{-l/2}^{l/2} dx \frac{1}{r^2} \sqrt{1 + \frac{r'(x)^2}{f(r)}} \quad (4.7)$$

where  $r'(x) \equiv \frac{dr(x)}{dx}$ . Using the standard procedure of minimization we obtain the profile of the minimal surface, given by

$$\frac{dr(x)}{dx} = \sqrt{f(r) \left( \frac{r_t^4}{r^4} - 1 \right)} \quad (4.8)$$

where  $r_t$  is the turning point of the minimal surface in the bulk. Using the expression for  $r'(x)$  we obtain the area of the minimal surface in the bulk and the length of the subsystem lying at the boundary. They are

$$A = 2L \int_{r_c}^{r_t} dr \frac{1}{r^2 \sqrt{f(r)(1 - r^4/r_t^4)}} ; \quad l = 2 \int_0^{r_t} dr \frac{(r/r_t)^2}{\sqrt{f(r)(1 - r^4/r_t^4)}} \quad (4.9)$$

where  $r_c$  is the ultra-violet cutoff. To make our computation easier we choose a new coordinate  $u = r/r_t$ . In this new coordinate, we have  $f(u) = 1 - u^2/u_0^2$ , where  $u_0 = r_h/r_t$  and the scaled UV cutoff  $\delta$  is defined as  $\delta = r_c/r_t$ . Therefore, the expressions for the subsystem length and minimal hypersurface area in the new coordinate are given by

$$l = 2r_t \int_0^1 du \frac{u^2}{\sqrt{(1 - u^4)f(u)}} \quad (4.10)$$

$$A = \frac{2L}{r_t} \int_\delta^1 du \frac{1}{u^2 \sqrt{(1 - u^4)f(u)}} . \quad (4.11)$$

## 4.2 Holographic Entanglement Entropy

We are now ready to compute the holographic entanglement entropy for the Lifshitz black hole. The explicit computation of entanglement entropy requires evaluation of the integrals (4.10) and (4.11) which contains a term  $1/\sqrt{f(u)}$  involving the lapse function. It is to be noted that for a  $(3 + 1)$ -dimensional pure Lifshitz spacetime the lapse function  $f(u)$  is equal to 1 [63, 46]. In that case, evaluation of integrals of the form (4.10) and (4.11) are trivial. As in our case,

$f(u) = 1 - u^2/u_0^2$ , computation of the integrals become non-trivial. However, they can be evaluated analytically if we expand  $1/\sqrt{f(u)}$  binomially as

$$\frac{1}{\sqrt{f(u)}} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n + 1)} u^n. \quad (4.12)$$

Using the above expression in eq.(4.10) we get the subsystem length, which reads

$$\frac{l}{r_t} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} + \frac{3}{4})}{2\Gamma(n + 1)\Gamma(\frac{n}{2} + \frac{5}{4})} \left(\frac{r_t}{r_h}\right)^{2n}. \quad (4.13)$$

As the above expression contains infinite number of terms, a few comments on its divergence structure is required. When  $r_t \ll r_h$  then we can terminate the series for some value of  $n$  as the higher order terms have negligible contribution. This implies that the subsystem length  $l$  is small, that is,  $l/r_h \ll 1$  with  $r_h$  kept fixed. However, when the turning point approaches the black hole horizon ( $r_t \rightarrow r_h$ ), one cannot terminate the series unlike the previous case. Then we have to look at the divergence structure of the expression in eq.(4.13). It is evident that the expression for subsystem length  $l$  goes as  $\sim \frac{1}{n} \left(\frac{r_t}{r_h}\right)^{2n}$  for large values of  $n$ . We rewrite eq.(4.13) in the following way

$$\frac{l}{r_t} = \frac{\sqrt{\pi}\Gamma(3/4)}{2\Gamma(5/4)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} + \frac{3}{4})}{2\Gamma(n + 1)\Gamma(\frac{n}{2} + \frac{5}{4})} \left(\frac{r_t}{r_h}\right)^{2n}. \quad (4.14)$$

The second term goes as  $\sim \frac{1}{2\sqrt{2}n}$  for large values of  $n$ . Therefore the series is divergent in  $r_t \rightarrow r_h$  limit (this can be seen by the comparison test). Now we add and subtract the divergent part to rewrite the above expression as

$$\frac{l}{r_t} = \frac{\sqrt{\pi}\Gamma(3/4)}{2\Gamma(5/4)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} + \frac{3}{4})}{2\Gamma(n + 1)\Gamma(\frac{n}{2} + \frac{5}{4})} \left(1 - \frac{1}{\sqrt{2}n}\right) \left(\frac{r_t}{r_h}\right)^{2n} - \frac{1}{\sqrt{2}} \log(1 - r_t^2/r_h^2). \quad (4.15)$$

In the above expression, the divergent piece  $\frac{1}{\sqrt{2}} \log(1 - r_t^2/r_h^2)$  has been separated out. It is important to note that, though we are considering  $r_t \rightarrow r_h$ , the hypersurface cannot penetrate the black hole horizon [69]. Therefore we can use the approximation  $r_t \simeq r_h(1 - \epsilon)$ , where  $\epsilon$  is very small, to rewrite eq.(4.15) as

$$\frac{l}{r_h} = k_1 - \frac{1}{\sqrt{2}} \log(2\epsilon) + \mathcal{O}(\epsilon) \quad (4.16)$$

with

$$k_1 = \frac{\sqrt{\pi}\Gamma(3/4)}{2\Gamma(5/4)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} + \frac{3}{4})}{2\Gamma(n+1)\Gamma(\frac{n}{2} + \frac{5}{4})} \left(1 - \frac{1}{\sqrt{2n}}\right). \quad (4.17)$$

We may now proceed to compute the minimal area of the hypersurface in the bulk. As we have to use the series form of the lapse factor (eq. (4.12)) in the area integral (eq. (4.11)), the expression for minimal area will contain infinite number of terms, given by

$$A = \frac{2L}{r_t} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)u_0^{2n}} \int_{\delta}^1 du \frac{u^{2n-2}}{\sqrt{1-u^4}}. \quad (4.18)$$

It is interesting to note that not all integrals in the above expression are convergent. The integral with  $n = 0$  is divergent in the  $\delta \rightarrow 0$  limit. Therefore, we perform the computation for the  $n = 0$  term separately and we get

$$A_{n=0} = \frac{2L}{r_t} \left( \frac{1}{\delta} - \frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} \right). \quad (4.19)$$

We observe that the above expression contains  $1/\delta$  term, which is divergent. This is obvious as  $A_{n=0}$  is the minimal area of the hypersurface for the pure Lifshitz spacetime, which should have a UV divergent term. Next we compute the  $A_{n \geq 1}$  terms and combine them with  $A_{n=0}$  to get an expression for minimal area as

$$A = \frac{2L}{r_t} \left( \frac{1}{\delta} - \frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} - \frac{1}{4})}{4\Gamma(n+1)\Gamma(\frac{n}{2} + \frac{1}{4})} \left(1 - \frac{4\Gamma(\frac{n}{2} + \frac{1}{4})\delta^{2n-1}}{\sqrt{\pi}\Gamma(\frac{n}{2} - \frac{1}{4})}\right) \left(\frac{r_t}{r_h}\right)^{2n} \right). \quad (4.20)$$

The above expression for minimal area contains an UV divergence term ( $\sim \frac{1}{\delta}$ ), which is exactly same to that of the  $(3+1)$ -dimensional  $AdS$  black brane. Therefore this UV divergence is a universal feature and does not depend on the nature of gravitational theory being relativistic or non-relativistic. We would like to point out that we have not made any approximation to find the expressions for subsystem length (4.13) and hypersurface area (4.20). Therefore the expressions are exact. As we are interested in the finite part of the minimal area, we subtract the divergent term proportional to  $1/\delta$  from eq.(4.20) to get

$$A_{finite} = \frac{2L}{r_t} \left( -\frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} - \frac{1}{4})}{4\Gamma(n+1)\Gamma(\frac{n}{2} + \frac{1}{4})} \left(\frac{r_t}{r_h}\right)^{2n} \right), \quad (4.21)$$

where we have taken the  $\delta \rightarrow 0$  limit to neglect  $\delta^{2n-1}$  term.

As it is obvious that the finite part of area of the minimal hypersurface ( $A_{finite}$ ) should depend on the length of the subsystem ( $l$ ), we now proceed to express  $A_{finite}$  in terms of  $l$ . For that we use the gamma function identity  $\Gamma(p+1) = p \Gamma(p)$  and the expression for the subsystem length (eq. (4.13)) to get

$$\begin{aligned} A_{finite} &= \frac{2L}{r_t} \left( -\frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \sum_{n=1}^{\infty} \frac{1}{4} \left( 1 + \frac{2}{2n-1} \right) \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{5}{4})} \left( \frac{r_t}{r_h} \right)^{2n} \right) \\ &= \frac{2L}{r_t} \left( -\frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \frac{l}{2r_t} + \sum_{n=1}^{\infty} \frac{1}{2(2n-1)} \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{5}{4})} \left( \frac{r_t}{r_h} \right)^{2n} \right) \end{aligned} \quad (4.22)$$

where we have used the gamma function identity  $\Gamma(p+1) = p \Gamma(p)$  in the first line and the expression for the subsystem length (eq. (4.13)) in the second line. For large values of  $n$  the third term in eq.(4.22) goes as  $\sim \frac{1}{2\sqrt{2}n^2} \left( \frac{r_t}{r_h} \right)^{2n}$ . Adding and subtracting  $\frac{1}{2\sqrt{2}n^2} \left( \frac{r_t}{r_h} \right)^{2n}$ , we can recast  $A_{finite}$  in the following way

$$\begin{aligned} A_{finite} &= \frac{2L}{r_t} \left( -\frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \frac{l}{2r_t} + \sum_{n=1}^{\infty} \left( \frac{1}{2(2n-1)} \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{5}{4})} - \frac{1}{2\sqrt{2}n^2} \right) \left( \frac{r_t}{r_h} \right)^{2n} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{1}{2\sqrt{2}n^2} \left( \frac{r_t}{r_h} \right)^{2n} \right) \\ &= \frac{2L}{r_t} \left( -\frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \frac{l}{2r_t} + \sum_{n=1}^{\infty} \left( \frac{1}{2(2n-1)} \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{5}{4})} - \frac{1}{2\sqrt{2}n^2} \right) \left( \frac{r_t}{r_h} \right)^{2n} \right. \\ &\quad \left. + \frac{1}{2\sqrt{2}} Li_2 \left[ \left( \frac{r_t}{r_h} \right)^2 \right] \right) \end{aligned} \quad (4.23)$$

This leads us to the expression for finite HEE

$$\begin{aligned} S_{finite} &= \frac{A_{finite}}{4G_{(4)}} \\ &= \frac{L}{2G_{(4)}r_t} \left( -\frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \frac{l}{2r_t} + \sum_{n=1}^{\infty} \left( \frac{1}{2(2n-1)} \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{5}{4})} - \frac{1}{2\sqrt{2}n^2} \right) \left( \frac{r_t}{r_h} \right)^{2n} \right. \\ &\quad \left. + \frac{1}{2\sqrt{2}} Li_2 \left[ \left( \frac{r_t}{r_h} \right)^2 \right] \right) \end{aligned} \quad (4.24)$$

We are now in a position to discuss the infrared (IR) behavior of HEE ( $S_{finite}$ ). But, before that it is important to know what do we mean by IR limit. It is evident from eq.(4.13) that if

we vary the length of the subsystem ( $l$ ), the turning point of the hypersurface ( $r_t$ ) also changes provided the radius of the horizon  $r_h$  is kept fixed. Therefore the turning point will be near to the horizon if the subsystem length is large ( $l/r_h \gg 1$ ). Physically this means that turning point lies in IR region and mathematically  $r_t/r_h \sim 1$ . This is what we mean by the IR limit. In this limit one can use the approximation  $r_t = r_h(1 - \epsilon)$  in the eq.(4.24) to get

$$\begin{aligned} S_{finite}^{(IR)} &= S_h + \frac{L}{2G_{(4)}r_h} (k_2 + k_3 \epsilon + k_4 \epsilon \log \epsilon) \\ &\simeq S_h + \frac{L}{2G_{(4)}r_h} k_2 + \mathcal{O}(\epsilon) \end{aligned} \quad (4.25)$$

where

$$S_h = \frac{lL}{4G_{(4)}r_h^2} \quad (4.26)$$

is the Bekenstein-Hawking entropy of the Lifshitz black hole and

$$\begin{aligned} k_2 &= -\frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \sum_{n=1}^{\infty} \left( \frac{1}{2(2n-1)} \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{5}{4})} - \frac{1}{2\sqrt{2}n^2} \right) + \frac{1}{2\sqrt{2}}\xi(2), \\ k_3 &= \frac{2(\log 2 - 1)}{2\sqrt{2}}; \quad k_4 = \frac{1}{\sqrt{2}} \end{aligned} \quad (4.27)$$

with  $\xi(2)$  being the Riemann zeta function. Thus, we find that the finite part of the holographic entanglement entropy in the IR limit is the black hole entropy (thermal entropy) plus correction terms. Since the black hole horizon radius is related to the black hole temperature ( $r_h \sim 1/\sqrt{T_h}$ ), the HEE in IR limit varies as

$$S_A^{finite} \sim T_h \left( 1 + c_1 \frac{1}{\sqrt{T_h}} \right) \quad (4.28)$$

with  $c_1$  being some numerical constant. This indicates that for high black hole temperature, the entanglement entropy transforms into the thermal entropy.

If on the other hand the turning point of the hypersurface lies near the boundary we say that the system is in ultra-violet (UV) regime. This is possible when the subsystem length  $l$  is small ( $l/r_h \ll 1$ ). Physically this implies that the the turning point  $r_t$  of the hypersurface will be far away from the black hole horizon  $r_h$  ( $r_t/r_h \ll 1$ ). This is what we mean by UV limit. In this limit we can take only first few terms of the binomial expansion of  $1/\sqrt{f(u)}$  (see eq.(4.12)).

Using this approximation we obtain the expression for finite part of HEE in UV limit, which is given by

$$S_{finite}^{(UV)} = \frac{L}{4G_{(4)}l} \left[ -4\pi \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2 + \frac{l^2}{r_h^2} \frac{1}{12} \left( \frac{\Gamma(1/4)}{\Gamma(3/4)} \right)^2 + \frac{l^4}{r_h^4} \frac{3}{80\pi} \left( \frac{\Gamma(1/4)}{\Gamma(3/4)} \right)^2 \left( 1 - \frac{5}{432} \left( \frac{\Gamma(1/4)}{\Gamma(3/4)} \right)^4 \right) \right]. \quad (4.29)$$

Thus the finite part of HEE in the UV limit depends on the subsystem size as  $S_{finite}^{(UV)} \sim \frac{1}{l} (\text{constant} + \mathcal{O}(l^2))$ . On the other hand the finite part of HEE for a (3+1)-dimensional  $SAdS$  black hole goes as  $S_{finite}^{(UV)} \sim \frac{1}{l} (\text{constant} + \mathcal{O}(l^3))$  [70]. We see that the leading contribution to the finite part of HEE depends identically on the subsystem size ( $\sim \frac{1}{l}$ ) for both the  $SAdS$  black hole and the Lifshitz black hole. However the difference occurs in the sub-leading contribution to the HEE. This happens due to the difference in lapse factors. For the Lifshitz black hole in our consideration the lapse factor is given by,  $f(r) = 1 - \frac{r^2}{r_h^2}$ , whereas, for the (3+1)-dimensional  $SAdS$  black hole it is given by,  $f(r) = 1 - \frac{r^3}{r_h^3}$ .

We now proceed to get an expression for HEE in the near horizon approximation. This approximation is important when someone deals with the high temperature black holes. In this case the black hole horizon approaches the turning point of the hypersurface ( $r_h \rightarrow r_t$ ), which means  $u_0 \sim 1$ . Therefore the integrals for subsystem length and area of the minimal hypersurface (eq.(s) (4.10) and (4.11)) receives a major contribution when  $u \sim 1$ . Therefore we make a Taylor series expansion of the lapse function about  $u \sim u_0$  to get the following,

$$\begin{aligned} f(u) &= f(u_0) + (u - u_0)f'(u_0) + \frac{(u - u_0)^2}{2}f''(u_0) + \dots \\ &\approx 2 \left( 1 - \frac{u}{u_0} \right) \end{aligned} \quad (4.30)$$

where the higher order terms are neglected as  $u - u_0 \ll 1$ . Using this approximated form of  $f(u)$  in eq.(4.10) we obtain

$$\frac{l}{r_t} = \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\Gamma(3/4)}{\Gamma(5/4)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n+3}{4})}{2\sqrt{2}\Gamma(n+1)\Gamma(\frac{n+5}{4})} \left( \frac{r_t}{r_h} \right)^n. \quad (4.31)$$

The second term in the above expression goes as  $\sim \frac{1}{n} \left( \frac{r_t}{r_h} \right)^n$  for large values of  $n$ . This term is divergent in the limit  $r_h \rightarrow r_t$  (by comparison test for infinite series). We add and subtract the

divergent term to separate out the divergent term and rewrite eq.(4.31) in the following way

$$\frac{l}{r_t} = \frac{\sqrt{\pi} \Gamma(3/4)}{2\sqrt{2} \Gamma(5/4)} + \sum_{n=1}^{\infty} \left( \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n+3}{4})}{2\sqrt{2} \Gamma(n+1)\Gamma(\frac{n+5}{4})} - \frac{1}{\sqrt{2} n} \right) \left( \frac{r_t}{r_h} \right)^n - \frac{1}{\sqrt{2}} \log \left( 1 - \frac{r_t}{r_h} \right). \quad (4.32)$$

As the hypersurface cannot penetrate the black hole horizon we can use the approximation  $r_t \simeq r_h(1 - \epsilon)$  [69]. Using this the subsystem length can be written as

$$\frac{l}{r_h} = c_1 - \frac{1}{\sqrt{2}} \log \epsilon + \mathcal{O}(\epsilon) \quad (4.33)$$

with

$$c_1 = \frac{\sqrt{\pi} \Gamma(3/4)}{2\sqrt{2} \Gamma(5/4)} + \sum_{n=1}^{\infty} \left( \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n+3}{4})}{2\sqrt{2} \Gamma(n+1)\Gamma(\frac{n+5}{4})} - \frac{1}{\sqrt{2} n} \right). \quad (4.34)$$

We now use the approximated form of  $f(u)$  (see eq.(4.30)) to compute the area integral under the near horizon approximation, which is as follows

$$A = \frac{L}{r_t} \sum_{n=0}^{\infty} \frac{\sqrt{2}\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \frac{1}{u_0^n} \int_{\delta}^1 du \frac{u^{n-2}}{\sqrt{1-u^4}}. \quad (4.35)$$

The above expression contains infinite numbers integrals out of which integrals corresponding to  $n = 0, 1$  are divergent. We compute the divergent integrals separately, which reads

$$\begin{aligned} A_{n=0} &= \frac{\sqrt{2}L}{r_t} \left( \frac{1}{\delta} - \frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} \right) \\ A_{n=1} &= \frac{L}{\sqrt{2}r_h} \left[ \int_{\delta}^1 du \frac{1}{u} + \sum_{m=1}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi}\Gamma(m+1)} \int_{\delta}^1 du u^{4m-1} \right] \\ &= \frac{L}{\sqrt{2}r_h} \left( -\log \delta + \frac{\log 4}{4} \right) \end{aligned} \quad (4.36)$$

where we have used the expansion

$$1/\sqrt{1-u^4} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} u^{4n}. \quad (4.37)$$

Now computing  $A_{n \geq 2}$  terms and then combining with  $A_{n=0}$  and  $A_{n=1}$  in eq.(4.35) we get

$$A = \frac{\sqrt{2}L}{r_t} \left( \frac{1}{\delta} - \frac{r_t}{2r_h} \log \delta - \frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \frac{r_t}{8r_h} \log 4 + \sum_{n=2}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n-1}{4})}{4\Gamma(n+1)\Gamma(\frac{n+1}{4})} \left( \frac{r_t}{r_h} \right)^n \right). \quad (4.38)$$

This expression for area in near horizon limit is different from the usual cases in the matter of UV cutoff dependent divergence. We observe that the the expression in eq.(4.38) contains

logarithmic divergence term in addition to the usual  $1/\delta$  divergence term. The appearance of the logarithmic divergence may be an artifact of the near horizon approximation [71]. Now using the formula  $\Gamma(p+1) = p \Gamma(p)$  and neglecting the UV divergent part, we can recast eq.(4.38) in the following manner

$$\begin{aligned}
A_{finite} &= \frac{\sqrt{2}L}{r_t} \left( -\frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \frac{r_t}{8r_h} \log 4 + \sum_{n=2}^{\infty} \left( 1 + \frac{2}{n-1} \right) \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n-1}{4})}{4\Gamma(n+1)\Gamma(\frac{n+1}{4})} \left( \frac{r_t}{r_h} \right)^n \right) \\
&= \frac{\sqrt{2}L}{r_t} \left( \frac{l}{\sqrt{2}r_t} - \frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} - \frac{r_t}{8r_h} (2 - \log 4) \right. \\
&\quad \left. + \sum_{n=2}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n-1}{4})}{2(n-1)\Gamma(n+1)\Gamma(\frac{n+1}{4})} \left( \frac{r_t}{r_h} \right)^n \right) \quad (4.39)
\end{aligned}$$

where in the last step we have used the result for subsystem length in near horizon approximation (see eq.(4.31)). For large values of  $n$ , the last term of the above equation goes as  $\sim \frac{1}{n^2} \left( \frac{r_t}{r_h} \right)^n$ . Using this fact we rewrite the expression for the finite part of area as

$$\begin{aligned}
A_{finite} &= \frac{Ll}{r_t^2} + \frac{\sqrt{2}L}{r_t} \left[ -\frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} - \left( \frac{5}{4} - \frac{\log 4}{8} \right) \frac{r_t}{r_h} + Li_2 \left[ \left( \frac{r_t}{r_h} \right) \right] \right. \\
&\quad \left. + \sum_{n=2}^{\infty} \left( \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n-1}{4})}{2(n-1)\Gamma(n+1)\Gamma(\frac{n+1}{4})} - \frac{1}{n^2} \right) \left( \frac{r_t}{r_h} \right)^n \right]. \quad (4.40)
\end{aligned}$$

Once again using the definition of HEE and the approximation  $r_t \simeq r_h(1-\epsilon)$  we finally obtain

$$\begin{aligned}
S_{finite} &= \frac{A_{finite}}{4G_{(4)}} \\
&= S_h + \frac{L}{2\sqrt{2}G_{(4)}r_h} (c_2 + c_3\epsilon + \epsilon \log \epsilon) \quad (4.41)
\end{aligned}$$

where

$$S_h = \frac{lL}{4G_{(4)}r_h^2} \quad (4.42)$$

is the black hole entropy and

$$\begin{aligned}
c_2 &= -\frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} - \left( \frac{5}{4} - \frac{\log 4}{8} \right) + \sum_{n=2}^{\infty} \left( \frac{1}{2(n-1)} \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{5}{4})} - \frac{1}{n^2} \right) + \xi(2), \\
c_3 &= \frac{1}{4} - \frac{\log 4}{8}. \quad (4.43)
\end{aligned}$$

Note the similarity between the eqs.(4.25) and (4.41). Therefore, the finite part of the HEE in IR limit and near horizon limit has the same form except the UV dependent divergence structure.

Further, this means that the finite part of the HEE in the near horizon approximation reveals the same temperature dependence as in IR limit (see eq.(4.28)).

### 4.3 Generalized temperature

We have already seen in the previous section that in the IR limit ( $r_t \rightarrow r_h$ ) the leading contribution to the HEE comes from the black hole entropy  $S_h$  (see eq.(4.25)). The black hole entropy is also termed as thermal entropy. However, due to the microscopic properties of the underlying quantum system the HEE gets quantum corrections as we move away from the IR limit. Thus we may ask for a quantity called the generalized temperature ( $T_g$ ) which reduces to the black hole temperature  $T_h$  in the IR limit. On the other hand the Lifshitz black hole satisfies the first law of black hole thermodynamics [63]

$$dE = T_h dS_h \tag{4.44}$$

leading to

$$E = \int T_h dS_h = T_h S_h / 2 \tag{4.45}$$

where the internal energy  $E$  is given by

$$E = \frac{lL}{16\pi G_{(4)} r_h^4}. \tag{4.46}$$

The relation is important since it relates the near horizon quantities ( $S_h, T_h$ ) to the near boundary quantity ( $E$ ).

Keeping the equation (4.45) in mind we define the generalized temperature in the following way

$$\frac{1}{T_g} = \frac{S_{REE}}{2E} ; \quad S_{REE} = S_A - S_A^{(0)} = \frac{1}{4G_{(4)}} (A - A^{(0)}) \tag{4.47}$$

where  $S_{REE}$  is the renormalized HEE of the Lifshitz black hole which is the difference of the HEE of Lifshitz black hole ( $S_A$ ) and that of the pure Lifshitz spacetime  $S_A^{(0)}$ . It is to be noted that though  $S_A$  and  $S_A^{(0)}$  includes UV divergence their difference  $S_{REE}$  is free from any UV

divergence. For pure Lifshitz spacetime the minimal hypersurface area and subsystem length are given by [38]

$$A_0 = \frac{2L}{\delta r_t^{(0)}} - \frac{4\pi L}{l} \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2; \quad \frac{l}{r_t^{(0)}} = \frac{\sqrt{\pi}\Gamma(3/4)}{2\Gamma(5/4)} \quad (4.48)$$

Using the above result along with eq.(4.20) and eq.(4.46), the generalized temperature reads

$$\frac{1}{T_g} = \frac{2\pi r_h^2}{l} \left[ \frac{4\pi}{l} \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2 - \frac{2\sqrt{\pi}\Gamma(3/4)}{r_t \Gamma(1/4)} + \frac{1}{2r_t} \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} - \frac{1}{4})}{4\Gamma(n+1)\Gamma(\frac{n}{2} + \frac{1}{4})} \left( \frac{r_t}{r_h} \right)^{2n} \right]. \quad (4.49)$$

This clearly shows that the generalized temperature ( $T_g$ ) is a function of the subsystem length  $l$  and the turning point of the hypersurface  $r_t$  (when  $r_h$  is fixed). But eq(s).(4.13) and (4.31) suggests that  $r_t$  itself depends on  $l$ . So we can consider that  $T_g$  is a function of  $l$  alone. Therefore the generalized temperature changes with the change in subsystem size. Let us now look at the behavior of the generalized temperature in the extreme limits. The generalized temperature in the infra-red limit ( $r_t \rightarrow r_h$ ) takes the form

$$\frac{1}{T_g} = \frac{1}{T_h} + \frac{r_h^3}{l} \alpha + \frac{r_h^4}{l^2} \beta \quad (4.50)$$

with

$$\begin{aligned} \alpha &= 2\pi \left( -\frac{4\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{n}{2} + \frac{3}{4})}{(2n-1)\Gamma(n+1)\Gamma(\frac{n}{2} + \frac{5}{4})} \right) \\ \beta &= 8\pi^2 \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2. \end{aligned} \quad (4.51)$$

It is evident from eq.(4.50) that in the large subsystem size limit ( $l/r_h \gg 1$ ) the generalized temperature reduces to the black hole temperature  $T_h$  (thermodynamic temperature). The sub-leading terms in eq.(4.50) are due to quantum entanglement. In the ultra violet limit ( $r_t/r_h \ll 1$ ), the generalized temperature takes the form

$$\frac{1}{T_g} = \frac{\pi r_h^2}{6} \left( \frac{\Gamma(1/4)}{\Gamma(3/4)} \right)^2 \left( 1 + \frac{l^2}{r_h^2} \gamma \right) \quad (4.52)$$

where

$$\gamma = \frac{9}{20\pi} - \frac{1}{192\pi} \left( \frac{\Gamma(1/4)}{\Gamma(3/4)} \right)^4. \quad (4.53)$$

Few observations are in order at this point. In the UV limit ( $l/r_h \rightarrow 0$  with  $r_h$  kept fixed), the generalized temperature depends on the subsystem size as  $\sim c + \frac{1}{l^2}$ . This

is in contrast with the observation made in [43] where the constant term is absent, although the  $1/l^2$  dependence is present. This may be due to the fact that the non-relativistic gravity dual considered in [43] is a hyperscaling violating black brane geometry which is different from our Lifshitz black hole. Moreover, we observe that the inverse of the generalized temperature ( $1/T_g$ ) does not reduce to zero when  $l/r_h = 0$  ( $r_h \neq 0$ ). This contradicts the results obtained in ([61, 62]), where the inverse of the generalized temperature reduces to zero when  $l/r_h = 0$  ( $r_h \neq 0$ ). However the studies made in ([61, 62]) considers relativistic gravity dual. Therefore our result  $\frac{1}{T_g} \neq 0$  when  $l/r_h = 0$  is completely a non-relativistic phenomena. This is an important observation.

Another important term in the studies of HEE is the “entanglement temperature” ( $T_{ent}$ ) which is the  $l \rightarrow 0$  limit of the generalized temperature ( $T_g$ ). Thus the expression for  $T_{ent}$  as obtained from eq.(4.52) is given by

$$T_{ent} = \frac{6}{\pi r_h^2} \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2. \quad (4.54)$$

The entanglement temperature arises due to the presence of quantum entanglement and has nothing to do with the macroscopic properties of the system. Our analysis confirms that the generalized temperature ( $T_g$ ) becomes the black hole temperature ( $T_h$ ) for large subsystem size as shown in figure 4.1.(a). Figure 4.1.(b) shows the variation of  $\frac{d\beta_g}{d\log l}$  with  $l$ . This has been done to study the flow of  $\beta_g = 1/T_g$  which helps us to characterize the thermal and quantum nature of the system. Figure 4.1.(b) has a maximum near a critical value  $\frac{l_c}{r_h} = 4.91$ . Below this value of the subsystem size, the system behaves as a quantum system whereas above this value, the system approaches a thermal system.

## 4.4 Summary

We have obtained the expression for holographic entanglement entropy for a  $(3+1)$ -dimensional Lifshitz black hole with  $z = 2$  which is a non-relativistic system. In the infrared limit, we find that the finite part of entanglement entropy becomes the thermal entropy even in the case of non-relativistic backgrounds. However, in the ultra-violet limit, there is a difference

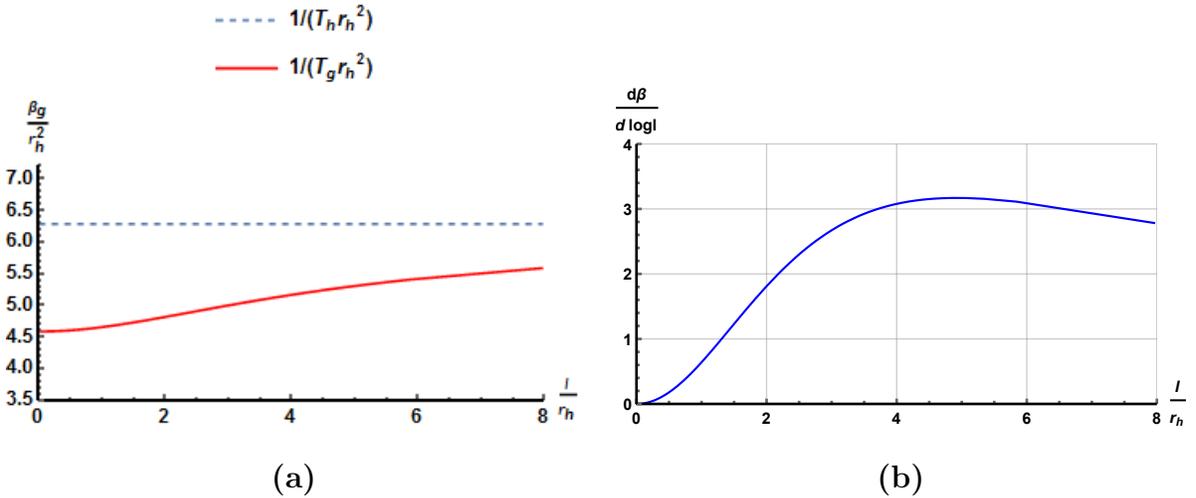


Figure 4.1: **(a)** Variation of  $\frac{\beta_g}{r_h^2} = \frac{1}{T_g r_h^2}$  with subsystem length, **(b)** Variation of  $\frac{d\beta_g}{d \log l}$  with subsystem length (the  $\frac{d\beta_g}{d \log l}$  axis has been scaled five times for visualization purpose).

in results between the relativistic and non-relativistic background. For  $(3 + 1)$ -dimensional  $SAdS$  black hole,  $S_{finite}^{(UV)} \sim \frac{1}{l} (constant + \mathcal{O}(l^3))$ , and for the Lifshitz black hole,  $S_{finite}^{(UV)} \sim \frac{1}{l} (constant + \mathcal{O}(l^2))$ . Therefore, the results differ in the subleading term. Further, we have computed the holographic entanglement entropy in the near horizon approximation. In this approximation, we see the presence of a logarithmic divergence involving the ultra-violet cutoff in addition to the usual  $1/\delta$  divergence. Then the notion of generalized temperature in terms of the renormalized entanglement entropy has been introduced. The generalized temperature ( $\frac{1}{T_g} = \frac{S_{REE}}{2E}$ ) has been defined on the line of the first law of black hole thermodynamics ( $E = T_h S_h/2$ ). It has been observed that the generalized temperature reduces to the black hole temperature (thermal temperature) for large subsystem size. We have also observed that the generalized temperature does not vanish when subsystem size becomes zero ( $l/r_h = 0$  but  $r_h \neq 0$ ). This is an important result as it does not have any counterpart in the relativistic background.

# Chapter 5

## Holographic entanglement entropy for a charged black hole in arbitrary dimension

This chapter is based on the work [39]. We have already seen in the introduction that the formula for holographic entanglement entropy (HEE) and the Bekenstein-Hawking formula for black hole entropy are very much similar. This led many to consider the entanglement entropy as the source of black hole entropy [16]-[22]. For this reason, the study of HEE for black holes is an interesting area of research. We know that anti-de Sitter (*AdS*) black holes are dual to a boundary conformal field theory (*CFT*) at finite temperature. Computation of HEE for such *AdS* black holes along with the behavior HEE in high and low temperature have been carried out in ([69],[72]-[74]). In [43], it was found that the difference in HEE of a thermally excited *AdS* spacetime and pure *AdS* spacetime ( $\Delta S_E$ ) is proportional to the change in energy ( $\Delta E$ ). This provides a relationship between the quantum information and the internal energy of the system. Moreover, a notion of entanglement temperature ( $T_{ent}$ ) was introduced which is different from the thermal temperature. A similar relation between the change in entanglement entropy ( $\Delta S_E$ ) and the change in energy ( $\Delta E$ ) was also obtained in [44], which was designated as the first law of entanglement thermodynamics. Entanglement thermodynamics has been explicitly studied

in different backgrounds including non-conformal and non-relativistic backgrounds ([42],[75]).

In [70], the HEE has been computed for a strip like subsystem lying in the boundary field theory which is dual to a charged *AdS* black hole in  $(3 + 1)$ - dimensions. The dependence of HEE on charge and temperature has been studied in an elaborative manner. Moreover, a relation like the first law of entanglement thermodynamics was obtained in the low temperature limit.

In this chapter, we extend the computation of HEE for a strip like subsystem in a charged black hole background to  $d$ -dimensions. We have explicitly computed the HEE in different temperature and charge limits of the black hole. Finally, a first law of thermodynamics like relation has been obtained in the small temperature limit.

## 5.1 Charged anti-de Sitter black hole

Depending upon the spin there exists two types of black holes, the Kerr-Newman black hole and the Reissner-Nordstrom black hole. The Kerr-Newman black hole has angular momentum and hence posses spin, whereas, the Reissner-Nordstrom (RN) black hole does not posses spin. We shall confine our activities of computing the HEE for RN black holes only.

We start by writing down the RN black hole metric with mass  $M$  and charge  $Q$  in *AdS* spacetime with planar horizon in  $d$ - dimensions. This reads [76]

$$\begin{aligned}
 ds^2 &= -\frac{r^2}{R^2}f(r)dt^2 + \frac{R^2dr^2}{r^2f(r)} + \frac{r^2}{R^2}(dx_1^2 + dx_2^2 + \dots + dx_{d-2}^2) \\
 f(r) &= 1 - \frac{M}{r^{d-1}} + \frac{Q^2}{r^{2(d-2)}}
 \end{aligned}
 \tag{5.1}$$

where  $R$  is the *AdS* length scale. For simplicity we chose  $R = 1$  for the rest of this chapter. The bulk theory is characterized by two parameters, namely, the mass  $M$  and the charge  $Q$ . However these two parameters are not independent, but related by the black hole horizon radius  $r_h$ . The black hole horizon  $r_h$  is given by  $f(r)|_{r=r_h} = 0$ . Using this we get

$$M = r_h^{d-1} \left( 1 + \frac{Q^2}{r_h^{2(d-2)}} \right).
 \tag{5.2}$$

Using the above relation we can rewrite the lapse function  $f(r)$  in terms of the black hole horizon  $r_h$  and the mass  $M$  in the following way

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{d-1} + Q^2 \left( \frac{1}{r^{2(d-2)}} - \frac{1}{r^{d-1}r_h^{d-3}} \right). \quad (5.3)$$

Therefore the bulk theory is now characterized by the charge and the black hole horizon.

Further, the Hawking temperature for this black hole is given by

$$T_H = \frac{r^2 f'(r)}{4\pi} \Big|_{r_h} = \frac{(d-1)r_h}{4\pi} \left[ 1 - \left(\frac{d-3}{d-1}\right) \frac{Q^2}{r_h^{2(d-2)}} \right]. \quad (5.4)$$

The expression for black hole temperature gives rise to an important relation between the black hole horizon radius  $r_h$  and the charge  $Q$  of the black hole

$$r_h^{2(d-2)} \geq \left(\frac{d-3}{d-1}\right) Q^2. \quad (5.5)$$

In the eq.(5.5), the equality sign holds for the extremal black hole (black hole with zero temperature) and the inequality holds for the non-extremal black hole (black hole with non zero temperature).

## 5.2 Computation of holographic entanglement entropy

We shall now setup the basic integrals to compute the HEE of the  $AdS$ -RN black hole. We consider a linear subsystem of width  $l$  at the boundary, given by

$$-\frac{l}{2} \leq x_1 \leq \frac{l}{2}; \quad 0 \leq x_2, x_3, \dots, x_{d-2} \leq L. \quad (5.6)$$

In order to get the entanglement entropy we have to compute the minimal area of the co-dimension two hypersurface in the bulk whose boundary coincides with the two ends of the linear subsystem situated at  $x_1 = -\frac{l}{2}$  and  $x_1 = \frac{l}{2}$  [24, 14]. This minimal area divided by  $4G_{(d)}$  is the entanglement entropy, where  $G_{(d)}$  is the Newton's gravitational constant in  $d$  dimensions.

Using the metric given in eq.(5.1) one can obtain the expression for area of the hypersurface, given by

$$\mathcal{A} = L^{d-3} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx_1 \sqrt{r^{2(d-2)} + \frac{(r')^2}{f(r)} r^{2(d-4)}}. \quad (5.7)$$

Note that we have parametrized the hypersurface as  $r = r(x_1)$  and  $r' = \frac{dr}{dx_1}$ . After using the standard procedure of minimization we get

$$\mathcal{A} = 2L^{d-3} \int_{r_t}^{\infty} \frac{r^{d-4} dr}{\sqrt{f(r) \left\{ 1 - \left( \frac{r_t}{r} \right)^{2d-4} \right\}}} \quad (5.8)$$

along with the profile of the minimal hypersurface characterized by

$$\frac{dr}{dx_1} = \sqrt{f(r) r^4 \left\{ \left( \frac{r^2}{r_t^2} \right)^{(d-2)} - 1 \right\}}. \quad (5.9)$$

In eq(s).(5.8) and (5.9), a term  $r_t$  has been introduced, which denotes the turning point of the extremal surface in the bulk, satisfying  $r'|_{r=r_t} = 0$ . The length of the subsystem ( $l$ ) can be obtained by integrating eq.(5.9)

$$\frac{l}{2} = \int_{r_t}^{\infty} \frac{r_t^{d-2} dr}{r^d \sqrt{f(r) \left\{ 1 - \left( \frac{r_t}{r} \right)^{2d-4} \right\}}} \quad (5.10)$$

It may be observed from eq.(5.8) that the area integral is divergent as we reach the boundary ( $r \rightarrow \infty$ ). Therefore we have to regularize the integral by introducing an infrared (IR) cutoff at  $r = r_b$ , where  $r_b$  assumes very large value. This IR cutoff in the bulk is holographically related to the ultraviolet (UV) cutoff  $a$  of the boundary field theory by the relation  $r_b = 1/a$ . The UV cutoff physically denotes lattice spacing in the boundary field theory. Due to the introduction of the cutoff, the expression for entanglement entropy would now contain a finite part and a cutoff dependent divergent part. We are interested in the finite part of the entanglement entropy as it can be used to study the high and low temperature (or charge) behavior of the entanglement entropy for the boundary field theory which is dual to the *AdS*-RN black hole.

In order to compute the integrals (5.8) and (5.10), we make a change of variable from  $r$  to  $u$  such that  $u = \frac{r_t}{r}$ . Therefore the lapse function now reads

$$f(u) = 1 - \left( \frac{r_h}{r_t} \right)^{d-1} u^{d-1} - \frac{Q^2}{r_h^{d-3}} \left( \frac{u}{r_t} \right)^{d-1} + Q^2 \left( \frac{u}{r_t} \right)^{2(d-2)}. \quad (5.11)$$

Further the length of the subsystem and the area of the hypersurface now read

$$l = \frac{2}{r_t} \int_0^1 du \frac{u^{d-2}}{\sqrt{1-u^{2(d-2)}}} \left( 1 - \left( \frac{r_h}{r_t} \right)^{d-1} u^{d-1} - \frac{Q^2}{r_h^{d-3}} \left( \frac{u}{r_t} \right)^{d-1} + Q^2 \left( \frac{u}{r_t} \right)^{2(d-2)} \right)^{-1/2} \quad (5.12)$$

$$\mathcal{A} = 2(Lr_t)^{d-3} \int_0^1 du \frac{u^{-(d-2)}}{\sqrt{1-u^{2(d-2)}}} \left( 1 - \left( \frac{r_h}{r_t} \right)^{d-1} u^{d-1} - \frac{Q^2}{r_h^{d-3}} \left( \frac{u}{r_t} \right)^{d-1} + Q^2 \left( \frac{u}{r_t} \right)^{2(d-2)} \right)^{-1/2} . \quad (5.13)$$

From eq.(5.12), it is evident that the subsystem length  $l$  would depend on the turning point  $r_t$ . So, we may invert the relation to express  $r_t$  in terms of  $l$ . As the area integral (see eq.(5.13)) also depends on  $r_t$ , we conclude that the area would depend upon subsystem length  $l$ , black charge  $Q$  and the horizon radius  $r_h$ . Moreover the state space for the boundary field theory dual to the *AdS*-RN black hole depends on two parameters, namely, the charge  $Q$  and the black hole temperature  $T_H$  (depends on  $r_h$  by eq.(5.4)). Therefore we have to consider the *AdS*-RN black hole in a particular ensemble. We chose it to be canonical ensemble with fixed black hole charge  $Q$ . Hence the expression for entanglement entropy would be a function of temperature ( $T_H$ ) only which would help us to analyze its high and low temperature behavior. With this basic expressions in hand, in the following sections we compute the HEE for both the extremal and non-extremal black hole.

### 5.2.1 Extremal black hole

Let us begin our analysis for HEE with the extremal black hole. Extremal black hole means that the black hole temperature is zero ( $T_H = 0$ ). This condition imposes a relation between the charge of the black hole and the horizon radius through the eq.(5.4), which is as follows

$$Q^2 = \left( \frac{d-1}{d-3} \right) r_h^{2(d-2)} . \quad (5.14)$$

The above equation implies that the charge may be small or large according to the black hole horizon radius. Therefore in the computation of HEE for an extremal black hole we may have two different limits - small and large charge limit. We shall compute the HEE in both the small and large charge limit. Now, using the eq.(5.14), we may recast the lapse function (see

eq.(5.11)) as

$$f(u) = 1 - \frac{2(d-2)}{d-3} \left(\frac{r_h u}{r_t}\right)^{d-1} + \frac{d-1}{d-3} \left(\frac{r_h u}{r_t}\right)^{2(d-2)}, \quad (5.15)$$

which in turn modify integrals (5.12),(5.13) as

$$l = \frac{2}{r_t} \int_0^1 du \frac{u^{d-2}}{\sqrt{1-u^{2(d-2)}}} \left(1 - \frac{2(d-2)}{d-3} \left(\frac{r_h u}{r_t}\right)^{d-1} + \frac{d-1}{d-3} \left(\frac{r_h u}{r_t}\right)^{2(d-2)}\right)^{-1/2} \quad (5.16)$$

$$\mathcal{A} = 2(Lr_t)^{d-3} \int_0^1 du \frac{u^{-(d-2)}}{\sqrt{1-u^{2(d-2)}}} \left(1 - \frac{2(d-2)}{d-3} \left(\frac{r_h u}{r_t}\right)^{d-1} + \frac{d-1}{d-3} \left(\frac{r_h u}{r_t}\right)^{2(d-2)}\right)^{-1/2}. \quad (5.17)$$

One should note that the above integrals do not have any known analytic solution. Therefore we have to use some approximation technique to evaluate those integrals. We shall use those technique in the following discussions.

**Small charge limit:** We are going to compute the HEE for an extremal *AdS*-RN black hole in small charge limit. This in turn means that we are dealing with a black hole of small horizon radius (see eq.5.14). To be more specific, in the small charge limit, we consider  $l \left(\frac{d-3}{d-1}\right)^{\frac{1}{2(d-2)}} Q^{\frac{1}{d-2}} \leq lr_h \ll 1$ . As the horizon radius  $r_h$  is very small, the black hole is well inside the bulk region and the turning point  $r_t$  of the hypersurface is far away from it. So in the Taylor series expansion of the lapse function we may neglect the contributions coming from the higher order terms of  $\left(\frac{r_h}{r_t}\right)$ . Thus

$$\frac{1}{\sqrt{f(u)}} \approx 1 + \frac{d-2}{d-3} \left(\frac{r_h u}{r_t}\right)^{d-1}. \quad (5.18)$$

Using the above expansion we may evaluate the integral in eq.(5.16) as

$$\begin{aligned} l &\approx \frac{2}{r_t} \int_0^1 \frac{u^{d-2} du}{\sqrt{1-u^{2(d-2)}}} \left(1 + \frac{d-2}{d-3} \left(\frac{r_h u}{r_t}\right)^{d-1}\right) \\ &= \frac{2}{r_t} \left[ \int_0^1 \frac{u^{d-2} du}{\sqrt{1-u^{2(d-2)}}} + \frac{d-2}{d-3} \left(\frac{r_h}{r_t}\right)^{d-1} \int_0^1 \frac{u^{2d-3} du}{\sqrt{1-u^{2(d-2)}}} \right] \\ &= \frac{2}{r_t} \left[ \sqrt{\pi} \frac{\Gamma\left(\frac{d-1}{2(d-2)}\right)}{\Gamma\left(\frac{1}{2(d-2)}\right)} + \frac{\sqrt{\pi}}{2(d-3)} \left(\frac{r_h}{r_t}\right)^{d-1} \frac{\Gamma\left(\frac{d-1}{d-2}\right)}{\Gamma\left(\frac{3d-4}{2(d-2)}\right)} \right]. \end{aligned} \quad (5.19)$$

This is the expression for subsystem length ( $l$ ) in terms of the turning point ( $r_t$ ). In order to get the expression for the turning point ( $r_t$ ) in terms of the subsystem length ( $l$ ) we have to invert the eq.(5.19). However the form of the eq.(5.19) suggests that we have to use perturbative approximation in terms of ( $lr_h$ ) to get

$$r_t = \frac{2}{l} \left[ \sqrt{\pi} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} + \frac{\sqrt{\pi}}{2(d-3)} \left( \frac{lr_h}{2} \right)^{d-1} \left( \frac{\Gamma(\frac{1}{2(d-2)})}{\sqrt{\pi}\Gamma(\frac{d-1}{2(d-2)})} \right)^{d-1} \frac{\Gamma(\frac{d-1}{d-2})}{\Gamma(\frac{3d-4}{2(d-2)})} \right]. \quad (5.20)$$

Now using same approximated form of lapse function (5.18), the area of the hypersurface (5.17) reads

$$\mathcal{A} = 2(Lr_t)^{d-3} \left[ \int_0^1 \frac{u^{-(d-2)} du}{\sqrt{1-u^{2(d-2)}}} + \frac{d-2}{d-3} \left( \frac{r_h}{r_t} \right)^{d-1} \int_0^1 \frac{u du}{\sqrt{1-u^{2(d-2)}}} \right]. \quad (5.21)$$

It is observed that the first integral in the above equation is the term for pure *AdS* spacetime which is divergent as  $u \rightarrow 0$ . In order to regularize the integral we replace the lower limit by the UV cut-off  $\frac{r_t}{r_b}$  and add a counter term ( $\frac{-2(Lr_b)^{d-3}}{d-3}$ ). This left us with the finite result

$$\begin{aligned} \mathcal{A}^{finite} &= 2(Lr_t)^{d-3} \left[ \int_{\frac{r_t}{r_b}}^1 \frac{u^{-(d-2)} du}{\sqrt{1-u^{2(d-2)}}} + \frac{d-2}{d-3} \left( \frac{r_h}{r_t} \right)^{d-1} \int_0^1 \frac{u du}{\sqrt{1-u^{2(d-2)}}} \right] - \frac{2(Lr_b)^{d-3}}{d-3} \\ &= 2(Lr_t)^{d-3} \left[ \frac{\sqrt{\pi}}{(d-2)} \frac{\Gamma(\frac{3-d}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} + \frac{\sqrt{\pi}}{(d-3)} \left( \frac{r_h}{r_t} \right)^{d-1} \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})} \right]. \end{aligned} \quad (5.22)$$

Thus we get  $\mathcal{A}^{finite}$  in terms of the turning point  $r_t$  of the hypersurface. Using eq.(5.20) we recast eq.(5.22) in terms of the subsystem length  $l$  as

$$\begin{aligned} \mathcal{A}^{finite} &= \left( \frac{L}{l} \right)^{d-3} \left[ -\frac{(2\sqrt{\pi})^{d-2}}{d-3} \left( \frac{\Gamma(\frac{d-1}{d-2})}{\Gamma(\frac{1}{2(d-2)})} \right)^{d-2} + \frac{d-2}{d(d-3)} \frac{(lr_h)^{d-1}}{4\sqrt{\pi}} \right. \\ &\quad \left. \times \left( \frac{\Gamma(\frac{1}{2(d-2)})}{\Gamma(\frac{d-1}{2(d-2)})} \right)^2 \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})} \right] \end{aligned} \quad (5.23)$$

where we have kept terms upto  $\mathcal{O}((lr_h)^{d-1})$ . The finite holographic entanglement entropy reads  $\left( \frac{\mathcal{A}^{finite}}{4G_{(d)}} \right)$  from the above equation

$$S_A^{finite} = S_A^{AdS} + S_A^{ext} \quad (5.24)$$

where

$$S_A^{AdS} = -\frac{(2\sqrt{\pi})^{d-2}}{4G_N^d(d-3)} \left(\frac{L}{l}\right)^{d-3} \left(\frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})}\right)^{d-2} \quad (5.25)$$

$$S_A^{ext} = \frac{L^{d-3}l^2r_h^{d-1}}{16\sqrt{\pi}G_{(d)}} \frac{d-2}{d(d-3)} \left(\frac{\Gamma(\frac{1}{2(d-2)})}{\Gamma(\frac{d-1}{2(d-2)})}\right)^2 \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})}. \quad (5.26)$$

Here  $S_A^{AdS}$  is the entanglement entropy for pure  $AdS$  spacetime [74].  $S_A^{AdS}$  is the leading contribution to the HEE and  $S_A^{ext}$  is the extra contribution to the HEE in small charge limit of the extremal black hole. Again we have a relation between the mass of the extremal black hole with its horizon radius

$$r_h^{d-1} = \frac{d-3}{2(d-2)} M^{ext}. \quad (5.27)$$

Using eq.(5.27) in eq.(5.26), we obtain

$$S_A^{ext} = kL^{d-3}l^2M^{ext} \quad (5.28)$$

where

$$k = \frac{1}{32dG_{(d)}\sqrt{\pi}} \left(\frac{\Gamma(\frac{1}{2(d-2)})}{\Gamma(\frac{d-1}{2(d-2)})}\right)^2 \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})}. \quad (5.29)$$

It is reassuring to note that our result as expressed in eq.(5.24) reduces to the result presented in [70] in the  $d = 4$  limit.

**Large charge limit:** In the large charge ( $Q$ ) limit the horizon ( $r_h$ ) of the black hole is also large by eq.(5.14). This in turn implies  $r_h l \gg 1$ . As the horizon radius is large we may consider that the horizon lies in close vicinity of the turning point ( $r_t$ ) of the hypersurface. Therefore the quantity  $\left(\frac{r_t}{r_h}\right) \sim 1$ . We know that the limit of the variable  $u$  in the area integral varies from  $\left(\frac{r_t}{r_b}\right)$  to 1, where  $r_b$  is the IR cutoff. Due to the presence of large charge we have large  $r_h$  and in turn large  $r_t$  as the hypersurface cannot penetrate the black hole horizon [69]. As both  $r_t$  and  $r_b$  are large, we may say  $\left(\frac{r_t}{r_b}\right) \sim 1$ . This means that the variable  $u = \frac{r_t}{r}$  remain close to  $u_0 = \frac{r_t}{r_h}$  during the integral. Therefore we may expand the lapse function  $f(u)$  around

$u = u_0$  as  $(u - u_0)$  is small enough. This expansion is called the near horizon expansion. Taylor expanding the lapse function in eq.(5.15) around  $u_0$  and neglecting higher order terms in  $(u - u_0)$  we get

$$\begin{aligned}
f(u) &= f(u_0) + f'(u_0)(u - u_0) + \frac{f''(u_0)}{2!}(u - u_0)^2 + \mathcal{O}((u - u_0)^3) \\
&= (d-1)(d-2) \left(1 - \frac{u}{u_0}\right)^2 + \mathcal{O}((u - u_0)^3) \\
&\simeq (d-1)(d-2) \left(1 - \frac{r_h u}{r_t}\right)^2.
\end{aligned} \tag{5.30}$$

This approximated form of  $f(u)$  may now be used to get the expression for the subsystem length and the area of the hypersurface. We start with the length of the subsystem which follows from eq.(5.16) as

$$l = \frac{2}{r_t \sqrt{(d-1)(d-2)}} \int_0^1 \frac{u^{d-2} du}{\sqrt{1 - u^{2(d-2)}}} \frac{1}{\left(1 - \frac{r_h}{r_t} u\right)}. \tag{5.31}$$

To get an analytic solution of the integral we make a binomial expansion of  $1/(1 - \frac{r_h}{r_t} u)$  and we obtain

$$\begin{aligned}
\frac{lr_t}{2} &= \frac{1}{\sqrt{(d-1)(d-2)}} \sum_{n=0}^{\infty} \left(\frac{r_h}{r_t}\right)^n \int_0^1 \frac{u^{n+d-2} du}{\sqrt{1 - u^{2(d-2)}}} \\
&= \frac{1}{2(d-2)^{3/2}} \sqrt{\frac{\pi}{d-1}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^n.
\end{aligned} \tag{5.32}$$

However, for large value of  $n$ , the above expression is divergent as  $\left(\frac{r_h}{r_t}\right) \sim 1$ . This can be checked easily. Using the Stirling formula and the properties of gamma function one can check that the above summation goes as  $\frac{\sqrt{2(d-2)}}{\sqrt{n}} \left(\frac{r_h}{r_t}\right)^n$  for large value of  $n$ . Therefore in  $\left(\frac{r_h}{r_t}\right) \sim 1$  limit the series is divergent by comparison test. Hence we isolate the divergent terms to get a finite expression for  $l$  as

$$\begin{aligned}
lr_t &= \frac{1}{(d-2)^{3/2}} \sqrt{\frac{\pi}{d-1}} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{2d-3}{2(d-2)})} + \sqrt{\frac{\pi}{(d-1)(d-2)}} \sum_{n=1}^{\infty} \left( \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{(d-2)\Gamma(\frac{n+2d-3}{2(d-2)})} - \sqrt{\frac{2}{(d-2)n}} \right) \left(\frac{r_h}{r_t}\right)^n \\
&\quad + \frac{1}{(d-2)} \sqrt{\frac{2\pi}{d-1}} Li_{\frac{1}{2}} \left(\frac{r_h}{r_t}\right)
\end{aligned} \tag{5.33}$$

where

$$Li_{\frac{1}{2}} \left(\frac{r_h}{r_t}\right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{r_h}{r_t}\right)^n \tag{5.34}$$

is polylogarithmic function. Now we consider  $r_t = r_h(1 + \epsilon)$ , where  $\epsilon$  is a very small positive number, as the hypersurface cannot penetrate the horizon of the black hole [69]. Using this approximation in eq.(5.33), we get

$$lr_h = k_1 + \sqrt{\frac{2}{d-1}} \left( \frac{\pi}{d-2} \right) \frac{1}{\sqrt{\epsilon}} + \mathcal{O}(\epsilon) \quad (5.35)$$

where

$$\begin{aligned} k_1 = & \sqrt{\frac{\pi}{d-1}} \frac{1}{(d-2)^{3/2}} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{2d-3}{2(d-2)})} + \sqrt{\frac{2\pi}{d-1}} \frac{1}{(d-2)} \zeta\left(\frac{1}{2}\right) \\ & + \sqrt{\frac{\pi}{(d-1)(d-2)}} \sum_{n=1}^{\infty} \left( \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{(d-2)\Gamma(\frac{n+2d-3}{2(d-2)})} - \sqrt{\frac{2}{(d-2)n}} \right). \end{aligned} \quad (5.36)$$

From eq.(5.35) we may also write

$$\epsilon \approx \frac{2\pi^2}{(d-1)(d-2)^2} \frac{1}{(lr_h - k_1)^2}. \quad (5.37)$$

One may also write an expression for  $\epsilon$  in terms of the black hole charge  $Q$  by using (5.14) to replace  $r_h$  in the above equation.

With the expression for subsystem length in hand we now proceed to evaluate the area integral in eq.(5.17). Using the approximated form of lapse function as in eq.(5.30) we may write the area integral as

$$\mathcal{A} = \frac{2(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \int_0^1 du \frac{u^{-(d-2)}}{\sqrt{1-u^{2(d-2)}}} \frac{1}{(1-\frac{r_h}{r_t}u)}. \quad (5.38)$$

Again we perform the binomial expansion to get an analytic solution of the integral as

$$\mathcal{A} = \frac{2(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \sum_{n=0}^{\infty} \left( \frac{r_h}{r_t} \right)^n \int_0^1 \frac{u^{n-d+2} du}{\sqrt{1-u^{2(d-2)}}}. \quad (5.39)$$

Careful examination of the above equation suggests that the integrals corresponding to  $n < (d-2)$  are divergent. To regularize the divergent terms corresponding to  $n = 0$  to  $n = d-3$  we introduce IR cut-off  $r_b$  in the integrals. Therefore the integral corresponding to  $n = 0$  is given by

$$\begin{aligned} \mathcal{A}_0^{finite} &= \frac{2(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \int_{\frac{r_t}{r_b}}^1 du \frac{1}{u^{d-2} \sqrt{1-u^{2(d-2)}}} - \frac{2(Lr_b)^{d-3}}{\sqrt{(d-1)(d-2)}} \\ &= -\frac{2\sqrt{\pi}(Lr_t)^{d-3}}{(d-3)\sqrt{(d-1)(d-2)}} \frac{\Gamma\left(\frac{d-1}{2(d-2)}\right)}{\Gamma\left(\frac{1}{2(d-2)}\right)}. \end{aligned} \quad (5.40)$$

The integral corresponding to  $n = 1$  is given by

$$\begin{aligned}\mathcal{A}_1^{finite} &= \frac{2(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \left(\frac{r_h}{r_t}\right) \int_0^1 du \frac{u^{-(d-3)}}{\sqrt{1-u^{2(d-2)}}} \\ &= \frac{2r_h L^{d-3} r_t^{d-4}}{\sqrt{(d-1)(d-2)}} \left[ \int_{\frac{r_t}{r_b}}^1 du \frac{1}{u^{d-3}} + \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)} \int_0^1 du u^{3-d+2k(d-2)} \right].\end{aligned}\quad (5.41)$$

In evaluating the above integral we have first used the following expansion

$$\frac{1}{\sqrt{1-y}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} y^n. \quad (5.42)$$

and then we have separated the first term (corresponding to  $k = 0$ ) as it is divergent. There is no contribution from the first integral since  $r_t$  and  $r_b$  are both large ( $r_t \sim r_b$ ). Hence the finite value of the integral corresponding to  $n = 1$  is given by

$$\mathcal{A}_1^{finite} = \frac{2r_h L^{d-3} r_t^{d-4}}{\sqrt{(d-1)(d-2)}} \left[ \frac{1}{d-4} + \frac{\sqrt{\pi}}{2(d-2)} \frac{\Gamma(\frac{4-d}{2(d-2)})}{\Gamma(\frac{2}{2(d-2)})} \right]. \quad (5.43)$$

In general the finite value of the integrals are

$$\mathcal{A}_m^{finite} = \frac{2(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \left(\frac{r_h}{r_t}\right)^m \left[ \frac{1}{d-m-3} + \frac{\sqrt{\pi}}{2(d-2)} \frac{\Gamma(\frac{m-d+3}{2(d-2)})}{\Gamma(\frac{m+1}{2(d-2)})} \right] \quad (5.44)$$

for  $m = 1, 2, \dots, (d-4)$ . We shall now compute the area integral for  $n = d-3$  in a detail

$$\begin{aligned}\mathcal{A}_{d-3} &= \frac{2(Lr_h)^{d-3}}{\sqrt{(d-1)(d-2)}} \int_0^1 du \frac{1}{u\sqrt{1-u^{2(d-2)}}} \\ &= \frac{2(Lr_h)^{d-3}}{\sqrt{(d-1)(d-2)}} \left[ \int_{\frac{r_t}{r_b}}^1 du \frac{1}{u} + \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)} \int_0^1 du u^{2k(d-2)-1} \right] \\ &= \frac{2(Lr_h)^{d-3}}{\sqrt{(d-1)(d-2)}} \left[ -\log\left(\frac{r_t}{r_b}\right) + \frac{\log 4}{2(d-2)} \right] \\ &\approx \frac{2(Lr_h)^{d-3}}{\sqrt{(d-1)(d-2)}} \frac{\log 4}{2(d-2)}\end{aligned}\quad (5.45)$$

where we have again used the expansion eq.(5.42) and separated out the integral corresponding to  $k = 0$ . The remaining area integrals in eq.(5.39) corresponding to  $n \geq (d-2)$  are given by

$$\begin{aligned}\mathcal{A}_{n \geq (d-2)} &= \frac{2(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \sum_{n=(d-2)}^{\infty} \left(\frac{r_h}{r_t}\right)^n \int_0^1 \frac{u^{n-d+2} du}{\sqrt{1-u^{2(d-2)}}} \\ &= \frac{2(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \sum_{n=(d-2)}^{\infty} \frac{\sqrt{\pi}}{2(d-2)} \frac{\Gamma(\frac{n-d+3}{2(d-2)})}{\Gamma(\frac{n+1}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^n.\end{aligned}\quad (5.46)$$

Again we find that the above contribution diverges as  $r_t$  approaches to  $r_h$ . For large values of  $n$  the factor inside the summation goes as  $\sim \frac{1}{\sqrt{n}} \left(\frac{r_h}{r_t}\right)^n$  and hence in the limit  $\frac{r_h}{r_t} \sim 1$  the sum is divergent by comparison test. We use the identity  $\Gamma(n+1) = n\Gamma(n)$  to rewrite eq.(5.46) as

$$\mathcal{A}_{n \geq (d-2)} = \frac{(Lr_t)^{d-3} \sqrt{\pi}}{\sqrt{(d-1)(d-2)}} \sum_{n=(d-2)}^{\infty} \left\{ \frac{1}{(d-2)} + \frac{1}{(n-d+3)} \right\} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^n. \quad (5.47)$$

Now for large value of  $n$ , the second term goes as  $\frac{\sqrt{2(d-2)}}{n\sqrt{n}} \left(\frac{r_h}{r_t}\right)^n$ . Therefore, it is convergent which can be checked by the comparison test. Using eq.(5.32) in the above equation we introduce the subsystem length ( $l$ ) in the expression for area as

$$\begin{aligned} \mathcal{A}_{n \geq (d-2)}^{finite} &= \frac{(Lr_t)^{d-3} \sqrt{\pi}}{\sqrt{(d-1)(d-2)}} \left[ \frac{\sqrt{(d-2)(d-1)}}{\sqrt{\pi}} l r_t - \sum_{m=0}^{d-3} \frac{\Gamma(\frac{m+d-1}{2(d-2)})}{(d-2)\Gamma(\frac{m+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^m \right. \\ &\quad \left. + \sum_{n=d-2}^{\infty} \frac{1}{(n-d+3)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^n \right]. \end{aligned} \quad (5.48)$$

In the above expression the leading contribution comes from the limit  $r_t = r_h$  and in this limit the second series in eq.(5.48) is convergent. We may now use  $r_t = r_h(1+\epsilon)$  to get the subleading terms in eq.(5.48) upto order  $\mathcal{O}(\epsilon)$ . However the second series in eq.(5.48) is not convergent at the subleading order. Hence we separate out the divergent terms of the series up to  $\mathcal{O}(\epsilon)$  and recast eq.(5.48) in the following way

$$\begin{aligned} \mathcal{A}_{n \geq (d-2)}^{finite} &= \frac{(Lr_t)^{d-3} \sqrt{\pi}}{\sqrt{(d-1)(d-2)}} \left[ \frac{\sqrt{(d-2)(d-1)}}{\sqrt{\pi}} l r_t - \sum_{m=0}^{d-3} \frac{\Gamma(\frac{m+d-1}{2(d-2)})}{(d-2)\Gamma(\frac{m+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^m \right. \\ &\quad \left. + \sum_{n=d-2}^{\infty} \left( \frac{1}{n-d+3} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} - \frac{\sqrt{2(d-2)}}{n\sqrt{n}} \right) \left(\frac{r_h}{r_t}\right)^n + \sum_{n=d-2}^{\infty} \frac{\sqrt{2(d-2)}}{n\sqrt{n}} \left(\frac{r_h}{r_t}\right)^n \right]. \end{aligned} \quad (5.49)$$

We can now write the expression for total area of the extremal surface by assembling all the

results together in eq.(5.39) in the following manner

$$\begin{aligned}
\mathcal{A}^{finite} &= \frac{(Lr_t)^{d-3}}{\sqrt{(d-1)(d-2)}} \left[ \sqrt{(d-2)(d-1)}lr_t - \frac{(d-2)\sqrt{\pi}}{d-3} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} \right. \\
&+ \sum_{n=1}^{d-4} \left( \frac{1}{d-n-3} + \frac{\sqrt{\pi}}{2(d-2)} \frac{\Gamma(\frac{n-d+3}{2(d-2)})}{\Gamma(\frac{n+1}{2(d-2)})} - \frac{\sqrt{\pi}}{2(d-2)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} \right) \left( \frac{r_h}{r_t} \right)^n \\
&+ \left( \frac{\log 4}{(d-2)} - \frac{\sqrt{\pi}}{(d-2)\Gamma(3/2)} \right) \left( \frac{r_h}{r_t} \right)^{d-3} \\
&+ \sqrt{\pi} \sum_{n=d-2}^{\infty} \left( \frac{1}{n-d+3} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} - \frac{\sqrt{2(d-2)}}{n\sqrt{n}} \right) \left( \frac{r_h}{r_t} \right)^n \\
&\left. \sqrt{2\pi(d-2)} \left( Li_{\frac{3}{2}} \left[ \frac{r_h}{r_t} \right] - \sum_{n=1}^{d-3} \frac{1}{n\sqrt{n}} \left( \frac{r_h}{r_t} \right)^n \right) \right] \quad (5.50)
\end{aligned}$$

where we have used the identity

$$\sum_{n=d-2}^{\infty} \frac{\sqrt{2(d-2)}}{n\sqrt{n}} \left( \frac{r_h}{r_t} \right)^n = \sqrt{2(d-2)} \left[ Li_{\frac{3}{2}} \left[ \frac{r_h}{r_t} \right] - \sum_{m=1}^{d-3} \frac{1}{m\sqrt{m}} \left( \frac{r_h}{r_t} \right)^m \right]. \quad (5.51)$$

Now we use the approximation  $r_t = r_h(1 + \epsilon)$  and simplify to finally obtain

$$\mathcal{A}^{finite} = L^{d-3}lr_h^{d-2} + (Lr_h)^{d-3} (K_1 + K_2\sqrt{\epsilon} + K_3\epsilon) + \mathcal{O}(\epsilon^{3/2}) \quad (5.52)$$

where

$$\begin{aligned}
K_1 &= \frac{1}{\sqrt{(d-1)(d-2)}} \left[ -2\sqrt{\pi} \frac{d-2}{d-3} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} + \frac{\log 4}{(d-2)} - \frac{\sqrt{\pi}}{(d-2)\Gamma(3/2)} \right. \\
&+ \sqrt{2\pi(d-2)} \left[ \zeta \left( \frac{3}{2} \right) - \sum_{n=1}^{d-3} \frac{1}{n\sqrt{n}} \right] + \sqrt{\pi} \sum_{n=d-2}^{\infty} \left[ \frac{1}{n-d+3} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} - \frac{\sqrt{2(d-2)}}{n\sqrt{n}} \right] \\
&\left. + \sum_{n=1}^{d-3} \left( \frac{1}{d-n-3} + \frac{d-2}{n+1} \frac{\Gamma(\frac{n-d+3}{2(d-2)})}{\Gamma(\frac{n+1}{2(d-2)})} \right) \right] \\
K_2 &= -\frac{2\sqrt{2}\pi}{\sqrt{d-1}} \\
K_3 &= \frac{2}{\sqrt{(d-1)(d-2)}} \left[ 1 + \frac{d}{d-1} \frac{\Gamma(\frac{-1}{2(d-2)})}{\Gamma(\frac{d-3}{2(d-2)})} + \sqrt{\frac{\pi(d-2)}{2}} \left( (d-1)\xi\left(\frac{3}{2}\right) - \xi\left(\frac{1}{2}\right) \right) \right]. \quad (5.53)
\end{aligned}$$

Thus the finite part of the HEE for extremal  $AdS$ -RN black hole in large charge regime is given by

$$S_A^{finite} = L^{d-3}lS_{BH}^{ext} + \frac{(Lr_h)^{d-3}}{4G_N^d} (K_1 + K_2\sqrt{\epsilon} + K_3\epsilon) + \mathcal{O}(\epsilon^{3/2}) \quad (5.54)$$

where  $S_{BH}^{ext} = \frac{r_h^{d-2}}{4G_N^d}$ .

### 5.2.2 Non-extremal black hole

In this section, our task is to compute the HEE for non-extremal  $AdS$ -RN black hole. Therefore we have non-zero Hawking temperature ( $T_H \neq 0$ ) which provides the following inequality

$$r_h^{2(d-2)} > \left(\frac{d-3}{d-1}\right) Q^2 \quad (5.55)$$

which means that the horizon radius  $r_h$  is bounded from below for a given black hole charge  $Q$ . Hence for small charge of the black hole the Hawking temperature may be low (for small horizon radius) or high (for large horizon radius) (see eq.(5.4)). On the other hand for large charge of the black hole the horizon radius can only be large, making a high Hawking temperature. Therefore, in this section we have to consider three different limits, they are, small charge & low temperature, small charge & high temperature and large charge with high temperature. Let us start with small charge with low temperature limit.

**Small Charge & low temperature limit :** As charge and temperature both are small, we are left with small black hole horizon radius (see eq.(s)(5.4) and (5.55)). This means that the turning point of the hypersurface is far away from the horizon and we may consider  $\frac{r_h}{r_t} \ll 1$ . For simplicity we introduce a new quantity  $\alpha = \frac{Q}{r_h^{d-2}}$  to recast the lapse function (see eq.(5.11) in the following way

$$f(u) = 1 - \left\{ 1 + \alpha^2 - \left(\frac{r_h u}{r_t}\right)^{d-3} \right\} \left(\frac{r_h u}{r_t}\right)^{d-1}. \quad (5.56)$$

The integrals for subsystem length (eq.(5.12)) and area (eq.(5.13)) contains the lapse factor as  $\frac{1}{\sqrt{f(u)}}$ . As  $\frac{r_h}{r_t} \ll 1$ , we may Taylor expand  $\frac{1}{\sqrt{f(u)}}$  and neglect the terms which are in higher order of  $(\frac{r_h}{r_t})$  to have

$$\frac{1}{\sqrt{f(u)}} = 1 + \frac{1 + \alpha^2}{2} \left(\frac{r_h u}{r_t}\right)^{d-1} + \mathcal{O}\left(\left(\frac{r_h u}{r_t}\right)^{2(d-2)}\right). \quad (5.57)$$

Using this approximated form of the lapse function in eq.(5.12) we compute the length of subsystem in terms of the turning point ( $r_t$ ) as

$$\begin{aligned}
l &= \frac{2}{r_t} \left[ \int_0^1 \frac{u^{d-2} du}{\sqrt{1-u^{2(d-2)}}} + \frac{1+\alpha^2}{2} \left( \frac{r_h}{r_t} \right)^{d-1} \int_0^1 \frac{u^{2d-3} du}{\sqrt{1-u^{2(d-2)}}} \right] \\
\Rightarrow l &= \frac{2}{r_t} \left[ \sqrt{\pi} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} + \frac{1+\alpha^2}{2} \left( \frac{r_h}{r_t} \right)^{d-1} \frac{\sqrt{\pi}}{2(d-2)^2} \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{3d-4}{2(d-2)})} \right]. \quad (5.58)
\end{aligned}$$

Using the perturbative technique we get an expression for the turning point in terms of the subsystem length

$$r_t = \frac{2\sqrt{\pi}}{l} \left[ \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} + \frac{1+\alpha^2}{(d-2)^2} \frac{(lr_h)^{d-1}}{2^{d+1}} \left( \frac{\Gamma(\frac{1}{2(d-2)})}{\sqrt{\pi}\Gamma(\frac{d-1}{2(d-2)})} \right)^{d-1} \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{3d-4}{2(d-2)})} \right]. \quad (5.59)$$

Note that the perturbative technique is possible as the turning point is far away from the horizon ( $\frac{r_h}{r_t} \ll 1$ ).

After substituting eq.(5.57) in eq.(5.13) the extremal surface area reads

$$\mathcal{A} = 2(Lr_t)^{d-3} \left[ \int_0^1 \frac{u^{-d+2} du}{\sqrt{1-u^{2(d-2)}}} + \frac{1+\alpha^2}{2} \left( \frac{r_h}{r_t} \right)^{d-1} \int_0^1 \frac{u du}{\sqrt{1-u^{2(d-2)}}} \right]. \quad (5.60)$$

Again the first integral is divergent as  $u \rightarrow 0$  and it is the term for pure  $AdS$ . In order to regularize the integral we introduce the UV cut-off  $\frac{r_t}{r_b}$  for the lower limit of the integral and add a counter term ( $\frac{-2(Lr_b)^{d-3}}{d-3}$ ). This provides us the finite part of the expression in eq.(5.60),

$$\begin{aligned}
\mathcal{A}^{finite} &= 2(Lr_t)^{d-3} \left[ \int_{\frac{r_t}{r_b}}^1 \frac{u^{-d+2} du}{\sqrt{1-u^{2(d-2)}}} + \frac{1+\alpha^2}{2} \left( \frac{r_h}{r_t} \right)^{d-1} \frac{\sqrt{\pi}}{2(d-2)} \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})} \right] - \frac{2(lr_b)^{d-3}}{d-3} \\
&= \frac{(Lr_t)^{d-3} \sqrt{\pi}}{d-2} \left[ \frac{\Gamma(\frac{3-d}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} + \frac{1+\alpha^2}{2} \left( \frac{r_h}{r_t} \right)^{d-1} \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})} \right]. \quad (5.61)
\end{aligned}$$

We use eq.(5.59) to recast  $\mathcal{A}^{finite}$  in terms of the subsystem length ( $l$ ). This yields

$$\begin{aligned}
\mathcal{A}^{finite} &= \left( \frac{L}{l} \right)^{d-3} \left[ -\frac{(2\sqrt{\pi})^{d-2}}{d-3} \left( \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} \right)^{d-2} + \frac{1+\alpha^2}{8\sqrt{\pi}d} (lr_h)^{d-1} \right. \\
&\quad \left. \times \left( \frac{\Gamma(\frac{1}{2(d-2)})}{\Gamma(\frac{d-1}{2(d-2)})} \right)^2 \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})} \right], \quad (5.62)
\end{aligned}$$

which can be used to get the HEE of the subsystem as

$$S_A^{finite} = \frac{\mathcal{A}^{finite}}{4G_N^d} = S_A^{AdS} + S_A^{non-ext} \quad (5.63)$$

where

$$S_A^{AdS} = -\frac{(2\sqrt{\pi})^{d-2}}{4G_N^d(d-3)} \left(\frac{L}{l}\right)^{d-3} \left(\frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})}\right)^{d-2} \quad (5.64)$$

$$S_A^{non-ext} = \frac{(1+\alpha^2)L^{d-3}l^2r_h^{d-1}}{32d\sqrt{\pi}G_{(d)}} \left(\frac{\Gamma(\frac{1}{2(d-2)})}{\Gamma(\frac{d-1}{2(d-2)})}\right)^2 \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})}. \quad (5.65)$$

However we have an expression (eq.(5.2)) which connects the mass of non-extremal black hole with  $\alpha$  as,  $M = r_h^{d-1}(1+\alpha^2)$ . This may be used to represent  $S_A^{non-ext}$  in the following way

$$S_A^{non-ext} = kL^{d-3}l^2M^{non-ext} \quad (5.66)$$

where

$$k = \frac{1}{32dG_{(d)}\sqrt{\pi}} \left(\frac{\Gamma(\frac{1}{2(d-2)})}{\Gamma(\frac{d-1}{2(d-2)})}\right)^2 \frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})} \quad (5.67)$$

Furthermore, these expressions reproduce the results in [70] in the  $d=4$  limit.

**Small charge & high temperature limit :** We shall now check the behavior of the entanglement entropy in the high temperature limit when the  $AdS$ -RN black hole has small charge. The expression for Hawking temperature (eq.(5.4)) and the inequality (eq.(5.55)) for non-extremality suggests that in the small charge regime we may have high temperature only when the horizon radius ( $r_h$ ) is large. This implies that  $\frac{Q^2}{r_h^{2(d-2)}} \ll 1$ . To make our computation simpler, we introduce a new quantity  $\delta^2 = \frac{(d-3)Q^2}{(d-1)r_h^{2(d-2)}} \ll 1$  to recast the lapse function (see eq.(5.11)) as

$$f(u) = 1 - \left(\frac{r_h u}{r_t}\right)^{d-1} - \frac{d-1}{d-3} \left(\frac{r_h u}{r_t}\right)^{d-1} \delta^2 \left(1 - \left(\frac{r_h u}{r_t}\right)^{d-3}\right) \quad (5.68)$$

As  $\delta^2 \ll 1$ , we Taylor expand  $\frac{1}{\sqrt{f(u)}}$  around  $\delta=0$  and neglect the higher order terms to get

$$\frac{1}{\sqrt{f(u)}} \approx \frac{1}{\sqrt{1 - \left(\frac{r_h u}{r_t}\right)^{d-1}}} \left[ 1 + \frac{(d-1)\delta^2}{2(d-3)} \left(\frac{r_h u}{r_t}\right)^{d-1} \frac{1 - \left(\frac{r_h u}{r_t}\right)^{d-3}}{1 - \left(\frac{r_h u}{r_t}\right)^{d-1}} \right]. \quad (5.69)$$

This approximated form of lapse function can be used in eq.(5.12) to get the subsystem length as

$$l = \frac{2}{r_t} \left[ \int_0^1 \frac{u^{d-2} du}{\sqrt{1-u^{2(d-2)}}} \frac{1}{\sqrt{1-\left(\frac{r_h u}{r_t}\right)^{d-1}}} + \frac{(d-1)\delta^2}{2(d-3)} \left(\frac{r_h}{r_t}\right)^{d-1} \times \int_0^1 \frac{u^{2d-3} du}{\sqrt{1-u^{2(d-2)}}} \frac{\left(1-\left(\frac{r_h u}{r_t}\right)^{d-3}\right)}{\left(1-\left(\frac{r_h u}{r_t}\right)^{d-1}\right)^{3/2}} \right]. \quad (5.70)$$

It is to be noted that no known solutions are present in the literature for the above integrals. But we are interested in an analytic solution and this led us to use the following identities

$$\frac{1}{\sqrt{1-y}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)} y^n; \quad \frac{1}{(1-y)^{3/2}} = \sum_{n=0}^{\infty} \frac{2\Gamma(n+3/2)}{\sqrt{\pi}\Gamma(n+1)} y^n \quad (5.71)$$

in eq.(5.70) to express the integrals in a convenient form. We rewrite eq.(5.70) as

$$\begin{aligned} \frac{lr_t}{2} &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h}{r_t}\right)^{(d-1)n} \int_0^1 \frac{u^{(d-1)n+d-2} du}{\sqrt{1-u^{2(d-2)}}} \\ &+ \frac{(d-1)\delta^2}{2(d-3)} \left(\frac{r_h}{r_t}\right)^{d-1} \sum_{n=0}^{\infty} \frac{2\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h}{r_t}\right)^{(d-1)n} \int_0^1 \frac{u^{(d-1)n+2d-3} du}{\sqrt{1-u^{2(d-2)}}} \left(1-\left(\frac{r_h u}{r_t}\right)^{d-3}\right) \\ \Rightarrow lr_t &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{(d-2)\Gamma(n+1)} \frac{\Gamma\left(\frac{(d-1)n+d-1}{2(d-2)}\right)}{\Gamma\left(\frac{(d-1)n+2d-3}{2(d-2)}\right)} \left(\frac{r_h}{r_t}\right)^{(d-1)n} + \frac{(d-1)\delta^2}{d-3} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{(d-2)\Gamma(n+1)} \\ &\times \left(\frac{r_h}{r_t}\right)^{(d-1)n+d-1} \left[ \frac{\Gamma\left(\frac{(d-1)n+2d-2}{2(d-2)}\right)}{\Gamma\left(\frac{(d-1)n+3d-4}{2(d-2)}\right)} - \left(\frac{r_h}{r_t}\right)^{d-3} \frac{\Gamma\left(\frac{(d-1)n+3d-5}{2(d-2)}\right)}{\Gamma\left(\frac{(d-1)n+4d-7}{2(d-2)}\right)} \right]. \quad (5.72) \end{aligned}$$

Before we proceed, we need to look at the divergence structure of different terms of the above expression. We use some gamma function properties and Stirling approximation for large value of  $n$  to conclude that the first term goes as  $\sim \frac{1}{n} \left(\frac{r_h}{r_t}\right)^n$ , whereas the second and third term goes identically as  $\sim \left(\frac{r_h}{r_t}\right)^n$ . This means that the divergences in the second and third term cancels out each other and make our analysis simpler. Isolating the divergent terms we recast eq.(5.72)

in the following way

$$\begin{aligned}
lr_t &= \frac{\sqrt{\pi}}{d-2} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{2d-3}{2(d-2)})} + \sum_{n=1}^{\infty} \left[ \frac{\Gamma(n+\frac{1}{2})}{(d-2)\Gamma(n+1)} \frac{\Gamma(\frac{(d-1)n+d-1}{2(d-2)})}{\Gamma(\frac{(d-1)n+2d-3}{2(d-2)})} - \sqrt{\frac{2}{(d-2)(d-1)}} \frac{1}{n} \right] \left(\frac{r_h}{r_t}\right)^{(d-1)n} \\
&+ \frac{(d-1)\delta^2}{(d-3)(d-2)} \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{(d-1)n+2d-2}{2(d-2)})}{\Gamma(\frac{(d-1)n+3d-4}{2(d-2)})} - \sqrt{\frac{2(d-2)}{(d-1)}} \right] \left(\frac{r_h}{r_t}\right)^{(d-1)n+d-1} \\
&- \frac{(d-1)\delta^2}{(d-3)(d-2)} \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{(d-1)n+3d-5}{2(d-2)})}{\Gamma(\frac{(d-1)n+4d-7}{2(d-2)})} - \sqrt{\frac{2(d-2)}{(d-1)}} \right] \left(\frac{r_h}{r_t}\right)^{(d-1)n+2d-4} \\
&+ \frac{\delta^2}{d-3} \sqrt{\frac{2(d-1)}{d-2}} \frac{\left(1 - \left(\frac{r_h}{r_t}\right)^{d-3}\right)}{\left(1 - \left(\frac{r_h}{r_t}\right)^{d-1}\right)} \left(\frac{r_h}{r_t}\right)^{d-1} - \sqrt{\frac{2}{(d-1)(d-2)}} \log\left(1 - \left(\frac{r_h}{r_t}\right)^{d-1}\right) \quad (5.73)
\end{aligned}$$

Now in small charge limit, high temperature is possible only with large value value of horizon radius ( $r_h$ ). So the horizon approaches the turning point of the hypersurface  $r_h \sim r_t$  and we may write  $r_t = (1 + \epsilon)r_h$  where  $\epsilon$  is a very small positive number. Hence eq.(5.73) can be expressed as

$$lr_h = -\sqrt{\frac{2}{(d-1)(d-2)}} \log((d-1)\epsilon) + C_1 + \delta^2 C_2 + \mathcal{O}(\epsilon) \quad (5.74)$$

where

$$\begin{aligned}
C_1 &= \frac{\sqrt{\pi}}{d-2} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{2d-3}{2(d-2)})} + \sum_{n=1}^{\infty} \left[ \frac{\Gamma(n+\frac{1}{2})}{(d-2)\Gamma(n+1)} \frac{\Gamma(\frac{(d-1)n+d-1}{2(d-2)})}{\Gamma(\frac{(d-1)n+2d-3}{2(d-2)})} - \sqrt{\frac{2}{(d-2)(d-1)}} \frac{1}{n} \right] \\
C_2 &= \sqrt{\frac{2}{(d-2)(d-1)}} + \frac{(d-1)}{(d-3)(d-2)} \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{(d-1)n+2d-2}{2(d-2)})}{\Gamma(\frac{(d-1)n+3d-4}{2(d-2)})} - \sqrt{\frac{2(d-2)}{(d-1)}} \right] \\
&- \frac{(d-1)\delta^2}{(d-3)(d-2)} \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{(d-1)n+3d-5}{2(d-2)})}{\Gamma(\frac{(d-1)n+4d-7}{2(d-2)})} - \sqrt{\frac{2(d-2)}{(d-1)}} \right]. \quad (5.75)
\end{aligned}$$

The Hawking temperature in terms of our newly defined quantity  $\delta$  takes the form

$$T_H = \frac{(d-1)r_h}{4\pi} (1 - \delta^2). \quad (5.76)$$

Using the above equation in the eq.(5.74) we replace  $r_h$  by the Hawking temperature. Then we simplify to have

$$\epsilon \approx \epsilon_{ent} e^{-\sqrt{\frac{d-2}{2(d-1)}} 4\pi T_H l (1+\delta^2)} \quad (5.77)$$

where  $\epsilon_{ent} = \frac{1}{d-1}e^{C_1+C_2\delta^2}$ .

After using the approximated form of the lapse function (eq.(5.69)) in the integral (eq.(5.13)), the surface area reads

$$\begin{aligned}
\mathcal{A} &= 2(Lr_t)^{d-3} \left[ \int_0^1 du \frac{u^{-d+2}}{\sqrt{1-u^{2(d-2)}}} \frac{1}{\sqrt{1-\left(\frac{r_h u}{r_t}\right)^{d-1}}} \right. \\
&\quad \left. + \frac{(d-1)\delta^2}{2(d-3)} \left(\frac{r_h}{r_t}\right)^{d-1} \int_0^1 du \frac{u \left(1 - \left(\frac{r_h u}{r_t}\right)^{d-3}\right)}{\sqrt{1-u^{2(d-2)}} \left(1 - \left(\frac{r_h u}{r_t}\right)^{d-1}\right)^{3/2}} \right] \\
&= 2(Lr_t)^{d-3} \left[ \int_0^1 du \frac{u^{-d+2}}{\sqrt{1-u^{2(d-2)}}} \left(1 + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h u}{r_t}\right)^{(d-1)n}\right) + \frac{(d-1)\delta^2}{2(d-3)} \left(\frac{r_h}{r_t}\right)^{d-1} \right. \\
&\quad \left. \times \int_0^1 du \frac{u \left(1 - \left(\frac{r_h u}{r_t}\right)^{d-3}\right)}{\sqrt{1-u^{2(d-2)}}} \left(\sum_{n=0}^{\infty} \frac{2\Gamma(n + \frac{3}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h u}{r_t}\right)^{(d-1)n}\right) \right].
\end{aligned} \tag{5.78}$$

Again we have used the identities in eq.(5.71) to evaluate the integrals analytically. As the earlier cases here also the first integral represents the area integral for pure AdS spacetime which is divergent near  $u \rightarrow 0$ . To get rid of this divergence, we have to introduce the UV

cut-off  $\frac{r_t}{r_b}$  and add a counter term to obtain a finite value of the area as

$$\begin{aligned}
\mathcal{A}^{finite} &= 2(Lr_t)^{d-3} \left[ \int_{\frac{r_t}{r_b}}^1 du \frac{u^{-d+2}}{\sqrt{1-u^{2(d-2)}}} + \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h}{r_t}\right)^{(d-1)n} \int_0^1 du \frac{u^{(d-1)n-d+2}}{\sqrt{1-u^{2(d-2)}}} \right. \\
&\quad \left. + \frac{(d-1)\delta^2}{2(d-3)} \sum_{n=0}^{\infty} \frac{2\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h u}{r_t}\right)^{(d-1)n+d-1} \int_0^1 du \frac{u^{(d-1)n+1}}{\sqrt{1-u^{2(d-2)}}} \left(1 - \left(\frac{r_h}{r_t}\right)^{d-3}\right) \right] \\
&\quad - \frac{2(Lr_b)^{d-3}}{d-3} \\
&= (Lr_t)^{d-3} \left[ \frac{\sqrt{\pi}}{d-2} \frac{\Gamma(\frac{3-d}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} + \sum_{n=1}^{\infty} \frac{1}{d-2} \frac{\Gamma(n+1/2)\Gamma(\frac{n+1+d(n-1)}{2(d-2)})}{\Gamma(n+1)\Gamma(\frac{n+1+nd}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^{(d-1)n} \right. \\
&\quad \left. + \frac{\delta^2(d-1)}{(d-2)(d-3)} \sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)\Gamma(\frac{2+n(d-1)}{2(d-2)})}{\Gamma(n+1)\Gamma(\frac{d+n(d-1)}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^{(n+1)(d-1)} \right. \\
&\quad \left. - \frac{\delta^2(d-1)}{(d-2)(d-3)} \sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)\Gamma(\frac{(d-1)(n+1)}{2(d-2)})}{\Gamma(n+1)\Gamma(\frac{n(d+1)+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^{n(d-1)+2(d-2)} \right] \tag{5.79}
\end{aligned}$$

$$\begin{aligned}
&= (Lr_t)^{d-3} \left[ \frac{\sqrt{\pi}}{d-2} \frac{\Gamma(\frac{3-d}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \frac{1}{d-2} \left(1 + \frac{d-2}{n+1+(n-1)(d-2)}\right) \frac{\Gamma(n+1/2)\Gamma(\frac{(n+1)(d-1)}{2(d-2)})}{\Gamma(n+1)\Gamma(\frac{n+1+(n+2)(d-2)}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^{(d-1)n} \right. \\
&\quad \left. + \frac{\delta^2(d-1)}{(d-2)(d-3)} \sum_{n=0}^{\infty} \left(1 + \frac{d-2}{2+n(d-1)}\right) \right. \\
&\quad \times \frac{\Gamma(n+3/2)\Gamma(\frac{(n+2)(d-1)}{2(d-2)})}{\Gamma(n+1)\Gamma(\frac{n+2+(n+3)(d-2)}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^{(n+1)(d-1)} \\
&\quad \left. - \frac{\delta^2(d-1)}{(d-2)(d-3)} \sum_{n=0}^{\infty} \left(1 + \frac{d-2}{n(d-1)+d-1}\right) \right. \\
&\quad \left. \times \frac{\Gamma(n+3/2)\Gamma(\frac{n+1+(n+3)(d-2)}{2(d-2)})}{\Gamma(n+1)\Gamma(\frac{n+1+(n+4)(d-2)}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^{n(d-1)+2(d-2)} \right] . \tag{5.80}
\end{aligned}$$

Now we can use eq.(5.72) to introduce the subsystem length in the expression for the surface

area as

$$\begin{aligned}
\mathcal{A}^{finite} = & (Lr_t)^{d-3} \left[ \frac{\sqrt{\pi}}{3-d} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{2d-3}{2(d-2)})} + lr_t \right. \\
& + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2}) \Gamma(\frac{(d-1)n+d-1}{2(d-2)}) \left(\frac{r_h}{r_t}\right)^{(d-1)n}}{((d-1)n - (d-3)) \Gamma(n+1) \Gamma(\frac{(d-1)n+2d-3}{2(d-2)})} + \frac{(d-1)\delta^2}{d-3} \\
& \times \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n+1)} \left\{ \frac{\Gamma(\frac{(d-1)n+2(d-1)}{2(d-2)}) \left(\frac{r_h}{r_t}\right)^{(d-1)n+d-1}}{((d-1)n+2) \Gamma(\frac{(d-1)n+3d-4}{2(d-2)})} \right. \\
& \left. - \frac{\Gamma(\frac{(d-1)n+3d-5}{2(d-2)}) \left(\frac{r_h}{r_t}\right)^{(d-1)n+2d-4}}{((d-1)n+d-1) \Gamma(\frac{(d-1)n+4d-7}{2(d-2)})} \right\} \left. \right]. \tag{5.81}
\end{aligned}$$

Now we have to look at the divergence structure of the above equation. Using the properties of gamma function and the Stirling approximation we see that the first summation term goes as  $\sim \frac{1}{n^2} \left(\frac{r_h}{r_t}\right)^n$ , where as the other summation terms goes as  $\sim \frac{1}{n} \left(\frac{r_h}{r_t}\right)^n$  for large  $n$ . We see that in small charge and high temperature limit eq.(5.81) receives its leading contribution when  $r_h = r_t$ . However, in this limit those summation terms are not divergent. But when we will substitute  $r_t = r_h(1 + \epsilon)$  in eq.(5.81) and expand binomially to get terms that are higher order in  $\epsilon$ , we will see that the summation terms are not convergent at the order  $\mathcal{O}(\epsilon)$ . We consider contribution to surface area only upto order  $\mathcal{O}(\epsilon)$  as  $\epsilon$  is likely to be a tiny quantity. We now substitute  $r_t = r_h(1 + \epsilon)$  and separate out the divergent terms to get an expression for the finite part of the hypersurface area as

$$\begin{aligned}
\mathcal{A}^{finite} = & L^{d-3} l r_h^{d-2} + L^{d-3} r_h^{d-3} (K_1 + \delta^2 K_2) \\
& + L^{d-3} r_h^{d-3} (K_3 \epsilon + \delta^2 (K_4 \epsilon + K_5 \epsilon \log \epsilon)) \tag{5.82}
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \frac{\sqrt{\pi}}{3-d} \frac{\Gamma\left(\frac{d-1}{2(d-2)}\right)}{\Gamma\left(\frac{2d-3}{2(d-2)}\right)} + \frac{\sqrt{2(d-2)}}{(d-1)^{\frac{3}{2}}} \xi(2) \\
&+ \sum_{n=1}^{\infty} \left( \frac{1}{((d-1)n+3-d)} \frac{\Gamma(n+1/2)\Gamma\left(\frac{(n+1)(d-1)}{2(d-2)}\right)}{\Gamma(n+1)\Gamma\left(\frac{(d-1)n+2d-3}{2(d-2)}\right)} - \frac{1}{n^2} \frac{\sqrt{2(d-2)}}{(d-1)^{\frac{3}{2}}} \right) \\
K_2 &= \left(\frac{d-1}{d-3}\right) \left[ \left( \frac{\Gamma\left(\frac{d-1}{d-2}\right)}{2\Gamma\left(\frac{3d-4}{2(d-2)}\right)} - \frac{\Gamma\left(\frac{3d-5}{2(d-2)}\right)}{(d-1)\Gamma\left(\frac{4d-7}{2(d-2)}\right)} \right) \Gamma(3/2) \right. \\
&+ \sum_{n=1}^{\infty} \left( \frac{1}{2+n(d-1)} \frac{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(\frac{(d-1)(n+2)}{2(d-2)}\right)}{\Gamma(n+1)\Gamma\left(\frac{(d-1)n+3d-4}{2(d-2)}\right)} - \frac{1}{n} \frac{\sqrt{2(d-2)}}{(d-1)^{\frac{3}{2}}} \right) \\
&+ \sum_{n=1}^{\infty} \left( \frac{1}{(n+1)(d-1)} \frac{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(\frac{(d-1)n+3d-5}{2(d-2)}\right)}{\Gamma(n+1)\Gamma\left(\frac{(d-1)n+3d-7}{2(d-2)}\right)} - \frac{1}{n} \frac{\sqrt{2(d-2)}}{(d-1)^{\frac{3}{2}}} \right) \\
K_3 &= \sqrt{\frac{2(d-2)}{d-1}} (\log(d-1) - 1) \\
K_4 &= \left(\frac{d-1}{d-3}\right) \left[ \frac{2(d-2)\Gamma\left(\frac{3d-5}{2(d-2)}\right)}{(d-1)\Gamma\left(\frac{4d-7}{2(d-2)}\right)} - \frac{(d-1)\Gamma\left(\frac{d-1}{d-2}\right)}{2\Gamma\left(\frac{3d-4}{2(d-2)}\right)} \right] \Gamma\left(\frac{3}{2}\right) - \sqrt{\frac{2(d-2)}{d-1}} \log(d-1) \\
K_5 &= -\sqrt{\frac{2(d-2)}{d-1}}. \tag{5.83}
\end{aligned}$$

Therefore the finite part of the HEE in small charge and high temperature limit is given by

$$\begin{aligned}
S_A^{finite} &= L^{d-3} l S_{BH} + \frac{(Lr_h)^{d-3}}{4G_N^d} (K_1 + \delta^2 K_2) \\
&+ \frac{L^{d-3} r_h^{d-3}}{4G_N^d} (K_3 \epsilon + \delta^2 (K_4 \epsilon + K_5 \epsilon \log \epsilon)) \tag{5.84}
\end{aligned}$$

where  $S_{BH} = \frac{r_h^{d-2}}{4G_N^d}$ .

**Large charge & high temperature limit:** The non-extremality condition (see eq.(5.55)) put a lower bound on the horizon radius ( $r_h$ ) which implies that the horizon radius is large for large charge of the black hole. Further, eq.(5.4) suggests that only high Hawking temperature ( $T_H$ ) is possible in large charge limit of the black hole. Therefore we will analyze the high

temperature behavior of the HEE for a boundary field theory dual to the  $AdS$ -RN black hole. By large horizon radius, we mean  $r_h l \gg 1$ . Therefore the horizon approaches the turning point of the hypersurface ( $\frac{r_t}{r_h} \sim 1$ ). Hence all the assumptions that we have made in the case of extremal black hole with large charge are also applicable in this case. We may therefore use the near horizon approximation to expand the lapse factor. Expanding the lapse function around  $u_0 = \frac{r_t}{r_h}$  and neglecting higher order terms in  $(u - u_0)$  we get

$$\begin{aligned} f(u) &= f(u_0) + f'(u_0)(u - u_0) + \mathcal{O}((u - u_0)^2) \\ &\approx \left[ (d-1) - \frac{(d-3)Q^2}{r_h^{2(d-2)}} \right] \left( 1 - \frac{u}{u_0} \right). \end{aligned} \quad (5.85)$$

The prefactor in the above equation has already appeared once in the expression for the Hawking temperature (see eq.(5.4)). To simplify we denote the prefactor by  $\sigma$ , so that  $\sigma = (d-1) - \frac{(d-3)Q^2}{r_h^{2(d-2)}}$ . In the high temperature limit  $\sigma \rightarrow (d-1)$  where as in the low temperature limit we have  $\sigma \rightarrow 0$ . Using the approximated form of the lapse function (eq.(5.85)) we write the integral for subsystem length as

$$\begin{aligned} l &= \frac{2}{r_t \sqrt{\sigma}} \int_0^1 \frac{u^{d-2} du}{\sqrt{1 - u^{2(d-2)}}} \frac{1}{\sqrt{1 - \frac{r_h u}{r_t}}} \\ \Rightarrow l r_t &= \frac{2}{\sqrt{\sigma}} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n+1)} \left( \frac{r_h}{r_t} \right)^n \int_0^1 \frac{u^{n+d-2} du}{\sqrt{1 - u^{2(d-2)}}} \\ &= \frac{1}{(d-2)\sqrt{\sigma}} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n+1)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} \left( \frac{r_h}{r_t} \right)^n \end{aligned} \quad (5.86)$$

where in the last line we have used the identity in eq.(5.71). For large value of  $n$ , the above summation goes as  $\frac{\sqrt{2}}{(d-2)n} \left( \frac{r_h}{r_t} \right)^n$ . This may be checked by using the gamma function properties and Stirling formula in large  $n$  limit. Therefore the series is divergent by comparison test in the limit  $r_t \rightarrow r_h$ . We isolate the divergent terms to get a finite value of the subsystem length as

$$\begin{aligned} l r_t &= \frac{\sqrt{\pi}}{(d-2)\sqrt{\sigma}} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{2d-3}{2(d-2)})} + \frac{1}{\sqrt{\sigma}} \sum_{n=1}^{\infty} \left( \frac{\Gamma(n + \frac{1}{2})}{(d-2)\Gamma(n+1)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} - \sqrt{\frac{2}{d-2}} \frac{1}{n} \right) \left( \frac{r_h}{r_t} \right)^n \\ &\quad - \sqrt{\frac{2}{(d-2)\sigma}} \log \left( 1 - \frac{r_h}{r_t} \right). \end{aligned} \quad (5.87)$$

Now we use the approximation  $r_t = r_h(1 + \epsilon)$  to finally obtain

$$\sqrt{\sigma}lr_h = -\sqrt{\frac{2}{d-2}}\log(\epsilon) + D_1 + \mathcal{O}(\epsilon) \quad (5.88)$$

where

$$D_1 = \frac{\sqrt{\pi}}{(d-2)} \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{2d-3}{2(d-2)})} + \sum_{n=1}^{\infty} \left\{ \frac{\Gamma(n + \frac{1}{2})}{(d-2)\Gamma(n+1)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} - \sqrt{\frac{2}{d-2}} \frac{1}{n} \right\}. \quad (5.89)$$

Further one can find from eq.(5.88),

$$\epsilon = \epsilon_{ent} e^{-\frac{(d-2)\sigma}{2}lr_h} \quad (5.90)$$

where  $\epsilon_{ent} = e^{\frac{d-2}{2}D_1}$ . We may now proceed to compute the extremal surface area using the approximated form of lapse function (see eq.(5.85)). The extremal area integral reads

$$\mathcal{A} = \frac{2(Lr_t)^{d-3}}{\sqrt{\sigma}} \int_0^1 du \frac{1}{u^{d-2}\sqrt{1-u^{2(d-2)}}} \frac{1}{\sqrt{\sigma}\left(1 - \frac{r_h}{r_t}u\right)}. \quad (5.91)$$

Using the identities presented in eq.(5.71) the above integral can be simplified as

$$\mathcal{A} = \frac{2(Lr_t)^{d-3}}{\sqrt{\sigma}} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h}{r_t}\right)^n \int_0^1 \frac{u^{n-d+2}du}{\sqrt{1-u^{2(d-2)}}}. \quad (5.92)$$

The integrals corresponding to  $n < (d-2)$  are divergent when  $u \rightarrow 0$ . To regularize we put an UV cutoff  $r_t/r_b$  for the lower limit of the integrals corresponding to  $n = 0$  up to  $n = d-3$ . To obtain the finite part of area integrals we use the same procedure as used in section (5.2.1). This gives us

$$\begin{aligned} \mathcal{A}_0^{finite} &= -\frac{2\sqrt{\pi}(Lr_t)^{d-3}}{(d-3)\sqrt{\sigma}} \frac{\Gamma\left(\frac{d-1}{2(d-2)}\right)}{\Gamma\left(\frac{1}{2(d-2)}\right)} \\ \mathcal{A}_m^{finite} &= \frac{2(Lr_t)^{d-3}}{\sqrt{\sigma}} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h}{r_t}\right)^m \left[ \frac{1}{d-m-3} + \frac{\sqrt{\pi}}{2(d-2)} \frac{\Gamma(\frac{m-d+3}{2(d-2)})}{\Gamma(\frac{m+1}{2(d-2)})} \right]; (m=1,2,\dots,d-4) \\ \mathcal{A}_{d-3}^{finite} &= \frac{2(Lr_t)^{d-3}\Gamma\left(\frac{2d-5}{2}\right)}{\sqrt{\pi\sigma}\Gamma(d-2)} \left(\frac{r_h}{r_t}\right)^{d-3} \frac{\log 4}{2(d-2)}. \end{aligned} \quad (5.93)$$

For  $n \geq (d-2)$ , we get

$$\begin{aligned}
\mathcal{A}_{n \geq (d-2)} &= \frac{2(Lr_t)^{d-3}}{\sqrt{\sigma}} \sum_{n=(d-2)}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \left(\frac{r_h}{r_t}\right)^n \int_0^1 \frac{u^{n-d+2} du}{\sqrt{1-u^{2(d-2)}}} \\
&= \frac{2(Lr_t)^{d-3}}{\sqrt{\sigma}} \sum_{n=(d-2)}^{\infty} \frac{1}{2(d-2)} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n-d+3}{2(d-2)})}{\Gamma(\frac{n+1}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^n \\
&= \frac{(Lr_t)^{d-3}}{\sqrt{\sigma}} \sum_{n=(d-2)}^{\infty} \left( \frac{1}{d-2} + \frac{1}{n-d+3} \right) \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^n. \quad (5.94)
\end{aligned}$$

In terms of the subsystem length (see eq.(5.86)) we can recast the above expression as

$$\begin{aligned}
\mathcal{A}_{n \geq (d-2)}^{finite} &= \frac{(Lr_t)^{d-3}}{\sqrt{\sigma}} \left[ \sqrt{\sigma} l r_t - \sum_{m=0}^{d-3} \frac{1}{d-2} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m+1)} \frac{\Gamma(\frac{m+d-1}{2(d-2)})}{\Gamma(\frac{m+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^m \right. \\
&\quad \left. + \sum_{n=d-2}^{\infty} \frac{1}{n-d+3} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^n \right]. \quad (5.95)
\end{aligned}$$

For large value of  $n$ , the third term in the above equation goes as  $\frac{\sqrt{2(d-2)}}{n^2} \left(\frac{r_h}{r_t}\right)^n$ . Hence in the limit  $r_t \rightarrow r_h$  the third term is finite. But as soon as one substitute  $r_t = r_h(1 + \epsilon)$ , one can see that the third term in eq.(5.95) is divergent at first order in  $\epsilon$ . Therefore we separate out the divergent terms to get only the finite part as

$$\begin{aligned}
\mathcal{A}_{n \geq (d-2)}^{finite} &= \frac{(Lr_t)^{d-3}}{\sqrt{\sigma}} \left[ \sqrt{\sigma} l r_t - \sum_{m=0}^{d-3} \frac{1}{d-2} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m+1)} \frac{\Gamma(\frac{m+d-1}{2(d-2)})}{\Gamma(\frac{m+2d-3}{2(d-2)})} \left(\frac{r_h}{r_t}\right)^m \right. \\
&\quad + \sum_{n=d-2}^{\infty} \left\{ \frac{1}{n-d+3} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(\frac{n+d-1}{2(d-2)})}{\Gamma(\frac{n+2d-3}{2(d-2)})} - \frac{\sqrt{2(d-2)}}{n^2} \right\} \left(\frac{r_h}{r_t}\right)^n \\
&\quad \left. + \sum_{n=d-2}^{\infty} \frac{\sqrt{2(d-2)}}{n^2} \left(\frac{r_h}{r_t}\right)^n \right]. \quad (5.96)
\end{aligned}$$

On the other hand

$$\sum_{n=d-2}^{\infty} \frac{\sqrt{2(d-2)}}{n^2} \left(\frac{r_h}{r_t}\right)^n = \sqrt{2(d-2)} \left[ Li_2 \left[ \frac{r_h}{r_t} \right] - \sum_{m=1}^{d-3} \frac{1}{m^2} \left(\frac{r_h}{r_t}\right)^m \right]. \quad (5.97)$$

Using the above identity we get the total finite surface area

$$\begin{aligned}
\mathcal{A}^{finite} = & \frac{(Lr_t)^{d-3}}{\sqrt{\sigma}} \left[ \sqrt{\sigma} l r_t - \frac{2\sqrt{\pi}(d-2)}{d-3} \frac{\Gamma\left(\frac{d-1}{2(d-2)}\right)}{\Gamma\left(\frac{1}{2(d-2)}\right)} \right. \\
& + \sum_{n=1}^{d-4} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \left( \frac{\sqrt{\pi}}{n+1} \frac{\Gamma\left(\frac{n-d+3}{2(d-2)}\right)}{\Gamma\left(\frac{n+1}{2(d-2)}\right)} + \frac{2}{[d-(n+3)]} \right) \left(\frac{r_h}{r_t}\right)^n \\
& + \frac{\Gamma\left(\frac{2d-5}{2}\right)}{(d-2)\sqrt{\pi}\Gamma(d-2)} (\log 4 - \Gamma(3/2)) \left(\frac{r_h}{r_t}\right)^{d-3} \\
& + \sqrt{2(d-2)} \left( Li_2 \left[ \frac{r_h}{r_t} \right] - \sum_{n=1}^{d-3} \frac{1}{n^2} \left(\frac{r_h}{r_t}\right)^n \right) \\
& \left. + \sum_{n=d-2}^{\infty} \left( \frac{1}{n-d+3} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma\left(\frac{n+d-1}{2(d-2)}\right)}{\Gamma\left(\frac{n+2d-3}{2(d-2)}\right)} - \frac{\sqrt{2(d-2)}}{n^2} \right) \left(\frac{r_h}{r_t}\right)^n \right].
\end{aligned} \tag{5.98}$$

Now we use the approximation  $r_t = r_h(1 + \epsilon)$  in eq.(5.98) and keep the sub-leading terms upto order  $\epsilon$ . After simplification we get the area of extremal surface to be

$$\mathcal{A}^{finite} = L^{d-3} l r_h^{d-2} + \frac{(Lr_h)^{d-3}}{\sqrt{\sigma}} \left\{ K'_1 + K'_2 \epsilon + \mathcal{O}(\epsilon^2) \right\} \tag{5.99}$$

where

$$\begin{aligned}
K'_1 = & -2\sqrt{\pi} \frac{d-2}{d-3} \frac{\Gamma\left(\frac{d-1}{2(d-2)}\right)}{\Gamma\left(\frac{1}{2(d-2)}\right)} + \sqrt{\frac{d-2}{2}} \xi(2) - \sqrt{\frac{d}{2}} \sum_{n=1}^{d-3} \frac{1}{n^2} \\
& + \sum_{n=1}^{d-4} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)} \left[ \frac{\sqrt{\pi}}{n+1} \frac{\Gamma\left(\frac{n-d+3}{2(d-2)}\right)}{\Gamma\left(\frac{n+1}{2(d-2)}\right)} + \frac{2}{d-(n+3)} \right] \\
& + \frac{\Gamma\left(\frac{2d-5}{2}\right)}{\sqrt{\pi}(d-2)\Gamma(d-2)} \left( \log 4 - \frac{\sqrt{\pi}}{\Gamma(3/2)} \right) \\
& + \sum_{n=d-2}^{\infty} \left[ \frac{1}{n-d+3} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \frac{\Gamma\left(\frac{n+d-1}{2(d-2)}\right)}{\Gamma\left(\frac{n+2d-3}{2(d-2)}\right)} - \frac{\sqrt{2(d-2)}}{n^2} \right] \\
K'_2 = & \frac{\Gamma\left(\frac{2d-7}{2}\right)}{\sqrt{\pi}\Gamma(d-3)} \left( 2 + \frac{\sqrt{\pi}\Gamma\left(\frac{-1}{2(d-2)}\right)}{(d-3)\Gamma\left(\frac{d-3}{2(d-2)}\right)} \right) - \sqrt{2(d-2)}.
\end{aligned} \tag{5.100}$$

Therefore, in the large charge regime for non-extremal  $AdS$ -RN black hole the renormalized HEE is given by

$$S_A^{finite} = L^{d-3} l S_{BH}^{ext} + \frac{(Lr_h)^{d-3}}{4G_N \sqrt{\sigma}} \left\{ K'_1 + K'_2 \epsilon + \mathcal{O}(\epsilon^2) \right\} \tag{5.101}$$

where  $S_{BH}^{ext} = \frac{r_h^{d-2}}{4G_N^d}$ .

### 5.3 Entanglement thermodynamics

In this section, we want to get a relation like the first law of entanglement thermodynamics. In order to obtain such relation we have to take the difference in holographic entanglement entropy between a ground state and an excited state. The holographic principle (*AdS/CFT*) suggests that we may consider the extremal ( $T_H = 0$ ) *AdS*-RN black hole with small charge as the dual to the ground state of the boundary field theory. On the other hand, the non-extremal ( $T_H \neq 0$ ) *AdS*-RN black hole (low temperature limit) with small charge may be considered as the excited state of the boundary field theory. The difference in entanglement entropy and energy of these two states are given by

$$\begin{aligned}\Delta S_A &= S_A^{non-ext} - S_A^{ext} = kL^{d-3}l^2(M^{non-ext} - M^{ext}), \\ \Delta E_A &= \int_A dx_1 dx_2 \dots dx_{d-3} T_{tt}^{temp \neq 0} - \int_A dx_1 dx_2 \dots dx_{d-3} T_{tt}^{temp=0} \\ &= \frac{d-2}{16\pi G_N^d} L^{d-3}l(M^{non-ext} - M^{ext}).\end{aligned}\tag{5.102}$$

We obtain the first law of thermodynamics like relation by combining  $\Delta S_A$  and  $\Delta E_A$  as

$$\Delta S_A = \frac{\Delta E_A}{T_{ent}}\tag{5.103}$$

with

$$T_{ent} = \frac{2(d-2)^2}{\sqrt{\pi}} \left( \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} \right)^2 \left[ \frac{1}{\frac{\Gamma(\frac{1}{d-2})}{\Gamma(\frac{d}{2(d-2)})} - \frac{\Gamma(\frac{d-1}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})}} \right].\tag{5.104}$$

$T_{ent}$  denotes the entanglement temperature and it is a dimension dependent quantity as can be observed from the above expression.

## 5.4 Summary

In this chapter, we have studied the holographic entanglement entropy in the context of the  $AdS$ -RN black hole in arbitrary dimension for a strip like subsystem. We see that the extremality condition for Hawking temperature put a lower bound on the horizon radius ( $r_h^{2(d-2)} \geq (\frac{d-3}{d-1}) Q^2$ ). As there are two parameters, namely, the charge ( $Q$ ) and the Hawking temperature ( $T_H$ ), which specify the state space of the boundary field theory dual to the  $AdS$ -RN black hole, we have to consider a particular ensemble for the black hole. We choose the ensemble to be canonical with a fixed black hole charge ( $Q$ ). We find that for an extremal black hole there can be small and large charge limits. There can be low and high temperature limit for the small charge regime of the non-extremal black hole, whereas there can be only large temperature limit for a non-extremal black hole in large charge regime. We have computed the entanglement entropy for a linear subsystem in all the above-mentioned regimes. We have considered the extremal  $AdS$ -RN black hole with small charge as the gravitational dual to the zero-temperature ground state of the boundary field theory. The non-extremal black hole in low temperature and low charge limit has been considered as the gravity dual of the excited state of the boundary field theory. In this context, we have obtained a first law of entanglement thermodynamics. The expression for entanglement temperature suggests that its dimension dependent.

# Chapter 6

## Quantum information theoretic quantities for a boosted black brane

In this chapter (based on the work [40]), we have computed the holographic subregion complexity (HSC) for a thin strip like region lying on the boundary field theory whose gravity dual is the  $(d + 1)$ -dimensional boosted black brane. The thin strip approximation assures that the bulk extension of the strip penetrates only the ultra-violet region of the bulk geometry. Therefore, the boosted black brane spacetime may be considered as the small perturbation around the pure  $AdS$  spacetime. With this assumption, we have computed the HSC up to first and second order in perturbation parameter.

The boosted black brane setup is useful in the holographic studies (gauge-gravity duality). In particular, they have been used to study the  $AdS/CFT$  correspondence [77, 78]. In the classical p-brane configuration with a pp-wave propagating along a direction in the world-volume, one has to distinguish two cases, depending upon whether or not the configuration is Bogomol'nyi-Prasad-Sommerfield (BPS) saturated. The effect of inclusion of the pp-wave in non-BPS case is locally equivalent to performing a Lorentz boost transformation along the direction of propagation of the wave. This justifies the nomenclature of a p-branes with pp-wave propagating along a direction in the world volume as the boosted p-branes. Furthermore, the equivalence is valid locally if the direction of propagation of pp-wave is wrapped on a

circle, on the other hand, the equivalence is valid globally if the direction of propagation is uncompactified. Moreover, the boosted black brane is holographically related to the uniformly boosted strongly coupled thermal plasma [79]. This helps to gain insight into the properties of quark-gluon plasma using the *AdS/CFT* correspondence. This justifies our choice of the background geometry.

It is well known that the first law of entanglement thermodynamics undergoes a modification due to the presence of gauge charge [42, 80]. In [80, 81], the modification has been obtained while working with the boosted black brane spacetime. They have computed the holographic entanglement entropy (HEE) for both, entangling strip parallel to the direction of the boost and the strip perpendicular to the direction of the boost. It was observed that the HEE in the perpendicular case is greater than the parallel case. The reason for this asymmetry is the difference in entanglement pressure in those two directions. In our analysis, we have found a similar kind of asymmetry in HSC for parallel and perpendicular cases. We have related the asymmetry in HSC with the asymmetry in HEE. This indicates the origin of asymmetry in HSC towards the difference in entanglement pressure in parallel and perpendicular directions. Further, this also indicates an interdependence between the HSC and the HEE.

Another reason for looking at the HSC, particularly the second-order change in perturbation parameter, is the proposal of computing the holographic Fisher information metric from the HSC [82]. In the quantum information theory, there exist two important notions of distance - Fisher information metric and the fidelity susceptibility. Further, the quantum information literature says they are the same for two infinitesimally close pure states [83] and other related cases [84]-[86]. In general cases, they are related [87].

In this chapter, we have computed the Fisher information metric from the relative entropy following the proposal in [88] for both the boosted and pure black brane. Then we have followed the proposal in [82] to compute the Fisher information metric from the second-order change in HSC. On the other hand, the holographic computation of the fidelity susceptibility involves the computation of the volume of the maximal time slice in *AdS* spacetime [89]. It was also generalized to include mixed states. We have used this prescription to compute the fidelity

susceptibility for the boosted black brane and pure black brane.

## 6.1 Holographic subregion complexity for $AdS$ spacetime

We begin with a review of the computation of HSC for a linear entangling region lying in a conformal field theory whose gravity dual is the  $AdS$  spacetime [90]. Considering the  $AdS$  radius  $R = 1$ , the the  $AdS$  metric in  $(d + 1)$ - dimensions is given by

$$ds^2 = \frac{-dt^2 + dx_1^2 + \cdots + dx_{d-1}^2 + dz^2}{z^2} . \quad (6.1)$$

The specification of the linear entangling surface is given by:  $(-\frac{l}{2} \leq x_1 \leq \frac{l}{2})$ ;  $0 \leq x_i \leq L_i$ . To compute the HSC, we have to figure out the volume under the RT-extremal surface whose boundaries coincides with the two ends of the strip. To do this we assume  $x_1 = x_1(z)$ . Then the area of the RT-surface is given by

$$A_{(0)} = 2V_{(d-2)} \int_0^{z_*^{(0)}} \frac{dz}{z^{d-1}} \sqrt{1 + x_1'(z)^2}, \quad (6.2)$$

where,  $V_{(d-2)} = L_2 L_3 \cdots L_{d-1}$  and  $z_*^{(0)}$  is the turning point of the extremal surface in the bulk. Minimization of the area functional provides a profile of the extremal surface, given by

$$x_1'(z) = \frac{1}{\sqrt{\left(\frac{z_*^{(0)}}{z}\right)^{2(d-1)} - 1}} . \quad (6.3)$$

Now the identification,  $x_1(0) = l/2$ , leads us to the subsystem length

$$\begin{aligned} \frac{l}{2} &= \int_0^{z_*^{(0)}} \frac{dz}{\sqrt{\left(\frac{z_*^{(0)}}{z}\right)^{2(d-1)} - 1}} \\ &= z_*^{(0)} \int_0^1 t^{d-1} \frac{dt}{\sqrt{1 - t^{2(d-1)}}} \\ &= z_*^{(0)} b_0 \end{aligned} \quad (6.4)$$

where  $t = \frac{z}{z_*^{(0)}}$ . We are now in a position to compute the volume under the RT-extremal surface in the bulk. The expression for volume is given by,

$$V_{(0)} = 2V_{(d-2)} \int_\delta^{z_*^{(0)}} \frac{dz}{z^d} \int_0^{x_1(z)} dx_1(z) \quad (6.5)$$

where we have introduced the UV cutoff  $\delta$  to regularize the integral. Now using Eq.(6.3), we can recast Eq.(6.5) in the following way

$$\begin{aligned}
V_{(0)} &= 2V_{(d-2)} \int_{\delta}^{z_*^{(0)}} \frac{dz}{z^d} \int_z^{z_*^{(0)}} \left(\frac{u}{z_*^{(0)}}\right)^{d-1} \frac{du}{\sqrt{1 - \left(\frac{u}{z_*^{(0)}}\right)^{2(d-1)}}} \\
&= \frac{V_{(d-2)}}{(d-1)} \frac{l}{\delta^{d-1}} - \frac{2^{d-2} \pi^{\frac{(d-1)}{2}}}{(d-1)^2} \left( \frac{\Gamma\left(\frac{d}{2d-2}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)} \right)^{d-3} \frac{V_{(d-2)}}{l^{d-2}}. \tag{6.6}
\end{aligned}$$

In the second line of the above equation, we have used Eq.(6.4) to express  $V_{(0)}$  in terms of the strip length  $l$ .

Therefore, the HSC for a strip like entangling region in  $AdS_{d+1}$  spacetime is given by

$$\begin{aligned}
C_{(0)} &= \frac{V_{(0)}}{8\pi G_{(d+1)}} \\
&= \frac{V_{(d-2)}}{8\pi G_{(d+1)}(d-1)} \frac{l}{\delta^{d-1}} - \frac{2^{d-2} \pi^{\frac{(d-1)}{2}}}{8\pi G_{(d+1)}(d-1)^2} \left( \frac{\Gamma\left(\frac{d}{2d-2}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)} \right)^{d-3} \frac{V_{(d-2)}}{l^{d-2}}. \tag{6.7}
\end{aligned}$$

Note that similar to the HEE [14], the first term in HSC is divergent (volume law), where as the other term is finite. With this result in hand we may now proceed for the HSC in boosted black brane background.

## 6.2 Holographic subregion complexity for boosted black brane

We assume that the boost is along  $y$  - direction and the  $AdS$  radius is set to one ( $R = 1$ ). With this assumption, the boosted black brane metric in  $(d + 1)$ - dimensional spacetime takes the form

$$ds^2 = \frac{1}{z^2} \left( -\frac{f dt^2}{K} + K(dy - \omega)^2 + dx_1^2 + \dots + dx_{d-2}^2 + \frac{dz^2}{f} \right) \tag{6.8}$$

with

$$f(z) = 1 - \frac{z^d}{z_0^d}; \quad K(z) = 1 + \beta^2 \gamma^2 \frac{z^d}{z_0^d}; \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \tag{6.9}$$

where,  $0 \leq \beta \leq 1$  is the boost parameter and  $z_0$  is the horizon of the black brane. The Kaluza-Klein one form  $\omega$  is given by

$$\omega = \beta^{-1} \left(1 - \frac{1}{K}\right) dt. \quad (6.10)$$

Note that the metric in Eq.(6.8) has an anisotropy due to the presence of boost along the  $y$  - direction. This motivates us to analyze the effect of anisotropy in HSC due to the presence of boost parameter. For this reason we compute the HSC with two different configurations: entangling strip parallel to the direction of boost ( $y$  - direction), and entangling strip perpendicular to the direction of boost ( $x$  - direction).

### 6.2.1 Strip parallel to the direction of boost

We consider a strip of length  $l$  (same as pure  $AdS$  case) along  $y$  - direction. The strip is specified as,  $-l/2 \leq y \leq l/2$  and  $0 \leq x^i \leq L_i$ , with  $L_i \gg l$ . Further, we take the length  $l$  to be small enough so that the hypersurface penetrates only the UV region of the bulk geometry. With the parametrization of the hypersurface  $y = y(z)$ , the area of the bulk extension is given by

$$A_{\parallel} = 2V_{(d-2)} \int_{\delta}^{z_{*}^{\parallel}} \frac{dy}{z^{d-1}} \sqrt{K(z) + \frac{(\partial_y z)^2}{f(z)}} \quad (6.11)$$

where,  $V_{(d-2)} \equiv L_1 L_2 L_3 \cdots L_{d-2}$ ,  $\delta$  is the UV cutoff, and  $z_{*}^{\parallel}$  is the turning point of the extremal surface inside the bulk. Moreover, we choose  $V_{(d-2)}$  to have the same value as it posses in the pure  $AdS$  case (see Eq.(6.7)).

Now to find the extremal surface, we use the standard process to minimize the area integral in Eq.(6.11). This provides a profile of the extremal surface, given by

$$\frac{dy}{dz} \equiv \left(\frac{z}{z_{*}^{\parallel}}\right)^{d-1} \frac{1}{\sqrt{f(z)K(z)} \sqrt{\frac{K(z)}{K_{*}} - \left(\frac{z}{z_{*}^{\parallel}}\right)^{2d-2}}} \quad (6.12)$$

where,  $K_{*} = K(z)|_{z=z_{*}^{\parallel}}$ . The identification  $y(0) = l/2$  leads to an expression connecting the subsystem length ( $l$ ) with the turning point ( $z_{*}^{\parallel}$ ), given by

$$\frac{l}{2} = \int_0^{z_{*}^{\parallel}} dz \left(\frac{z}{z_{*}^{\parallel}}\right)^{d-1} \frac{1}{\sqrt{f(z)K(z)} \sqrt{\frac{K(z)}{K_{*}} - \left(\frac{z}{z_{*}^{\parallel}}\right)^{2d-2}}}. \quad (6.13)$$

As the turning point lies in UV region ( $z_*^\parallel \ll z_0$ ), we may consider

$$\left(\frac{z_*^\parallel}{z_0}\right)^d \ll 1, \quad \beta^2 \gamma^2 \left(\frac{z_*^\parallel}{z_0}\right)^d \ll 1, \quad (6.14)$$

to evaluate the integral (Eq.(6.13)) by expanding it around the pure AdS background. Note that this is true for finite boost only. With this approximation, eq.(6.13) can be expanded as

$$\begin{aligned} \frac{l}{2} &= z_*^\parallel \int_0^1 dt t^{d-1} \frac{1}{\sqrt{R}} \left[ 1 + \frac{1}{2} p^d t^d - \frac{1}{2} q^d t^d + \frac{1}{2} q^d \frac{1-t^d}{R} + \dots \right] \\ &= z_*^\parallel \left( b_0 + \frac{1}{2} (p^d b_1 - q^d b_1 + q^d I_l) \right) + \dots, \end{aligned} \quad (6.15)$$

where,  $t = \frac{z}{z_*^\parallel}$ ,  $R \equiv 1 - t^{2d-2}$ ,  $p = \frac{z_*^\parallel}{z_0}$ ,  $q^d = \beta^2 \gamma^2 \left(\frac{z_*^\parallel}{z_0}\right)^d$  and the dots indicates higher order terms in  $\left(\frac{z_*^\parallel}{z_0}\right)^d$ . The values of  $b_0, b_1$ , and  $I_l$  are provided in the Appendix.

As we are keeping the strip length  $l$  same for the pure *AdS* spacetime and the boosted black brane spacetime, the turning points of the extremal surfaces will be different. In order to express the new turning point  $z_*^\parallel$  in terms of the turning point  $z_*^{(0)}$  of pure *AdS*, we use Eqs.(6.15) and (6.4) to get

$$\begin{aligned} z_*^\parallel &= \frac{l/2}{b_0 + \frac{1}{2} (p^d b_1 - q^d b_1 + q^d I_l)} \\ &\simeq \frac{z_*^{(0)}}{1 + \frac{1}{2} (\bar{p}^d \frac{b_1}{b_0} - \bar{q}^d \frac{b_1}{b_0} + \bar{q}^d \frac{I_l}{b_0})} \end{aligned} \quad (6.16)$$

where,  $\bar{q}^d = \beta^2 \gamma^2 \left(\frac{z_*^{(0)}}{z_0}\right)^d$  and  $\bar{p} = \frac{z_*^{(0)}}{z_0}$ . Note that we have kept terms only up to  $\left(\frac{z_*^{(0)}}{z_0}\right)^d$  under the thin strip approximation.

We may now proceed to find the volume under the RT- extremal surface, given by

$$\begin{aligned} V_\parallel &= 2V_{(d-2)} \int_\delta^{z_*^\parallel} \frac{dz}{z^d} \sqrt{\frac{K(z)}{f(z)}} \int_z^{z_*^\parallel} dz \left(\frac{u}{z_*^\parallel}\right)^{d-1} \frac{1}{\sqrt{f(u)K(u)} \sqrt{\frac{K(u)}{K_*} - \left(\frac{u}{z_*^\parallel}\right)^{2d-2}}} \\ &= \frac{2V_{(d-2)}}{z_*^{\parallel d-2}} \int_{\frac{\delta}{z_*^\parallel}}^1 \frac{dt}{t^d} \sqrt{\frac{K(t)}{f(t)}} \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)K(w)} \sqrt{\frac{K(w)}{K_*} - w^{2d-2}}}, \end{aligned} \quad (6.17)$$

where  $w = \frac{u}{z_*^\parallel}$ ,  $K(w) = 1 + (wq)^d$ , and  $f(w) = 1 - (tp)^d$ . For finite boost, we can use the approximation as given in Eq.(6.14) to expand the volume integral in Eq.(6.17) around the

pure *AdS* spacetime volume. Therefore, we may expand the functions  $(K, f)$  and keep terms up to linear order to recast Eq.(6.17) as

$$\begin{aligned}
V_{\parallel} &= \frac{2V_{(d-2)}}{z_*^{\parallel d-2}} \int_{z_*^{\parallel}}^1 \frac{dt}{t^d} \left( 1 + \frac{t^d}{2}(p^d + q^d) \right) \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)K(w)}\sqrt{\frac{K(w)}{K_*} - w^{2d-2}}} \\
&= \frac{2V_{(d-2)}}{z_*^{\parallel d-2}} \int_{z_*^{\parallel}}^1 \frac{dt}{t^d} \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)K(w)}\sqrt{\frac{K(w)}{K_*} - w^{2d-2}}} \\
&\quad + \frac{V_{(d-2)}}{z_*^{\parallel d-2}} (p^d + q^d) \int_{z_*^{\parallel}}^1 dt \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)K(w)}\sqrt{\frac{K(w)}{K_*} - w^{2d-2}}} .
\end{aligned} \tag{6.18}$$

After evaluating above integrals, we use Eq.(6.6) and Eq.(6.16) to express  $V_{\parallel}$  in terms of the pure *AdS* volume  $V_{(0)}$ , as

$$\begin{aligned}
V_{\parallel} &= V_{(0)} - \frac{V_{(d-2)}\bar{p}^d}{(d-1)z_*^{(0)d-2}} \left( \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2-d)c_0 \right) \\
&\quad - \frac{V_{(d-2)}\bar{q}^d}{(d-1)z_*^{(0)d-2}} \left( \frac{(d-2)\pi}{2(d-1)^2 b_0} \left( \frac{2b_1}{b_0} - 1 \right) + c_2 - c_0 d \right) .
\end{aligned} \tag{6.19}$$

Note that we have kept terms up to first order in  $\bar{p}^d$  and  $\bar{q}^d$ . Moreover, the change in volume due to the change in background geometry in terms of the strip length is given by

$$\begin{aligned}
\Delta V_{\parallel} &= V_{\parallel} - V_{(0)} \\
&= -\frac{V_{(d-2)}l^2}{4b_0^2(d-1)z_0^d} \left[ \left( \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2-d)c_0 \right) \right. \\
&\quad \left. + \beta^2 \gamma^2 \left( \frac{(d-2)\pi}{2(d-1)^2 b_0} \left( \frac{2b_1}{b_0} - 1 \right) + c_2 - c_0 d \right) \right] .
\end{aligned} \tag{6.20}$$

Therefore, the change in HSC for an entangling region parallel to the direction of boost is given by

$$\begin{aligned}
\Delta C_{\parallel}^{(1)} &\equiv \frac{\Delta V_{\parallel}}{8\pi G_{(d+1)}} \\
&= -\frac{V_{(d-2)}l^2}{32\pi G_{(d+1)}b_0^2(d-1)z_0^d} \\
&\quad \times \left[ \left( \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2-d)c_0 \right) + \beta^2 \gamma^2 \left( \frac{(d-2)\pi}{2(d-1)^2 b_0} \left( \frac{2b_1}{b_0} - 1 \right) + c_2 - c_0 d \right) \right] .
\end{aligned} \tag{6.21}$$

This result matches with that of pure *AdS* black brane in  $\beta = 0$  limit [90].

For a boosted black brane, the change in HEE for the two cases up to first order in perturbation parameter, namely, for a strip parallel and perpendicular to the direction of boost are given by [81],

$$\begin{aligned}\Delta S_{\parallel} &= \frac{V_{(d-2)}l^2b_1(d+1)}{32G_{(d+1)}b_0^2z_0^d} \left( \frac{d-1}{d+1} + \frac{2}{d+1}\beta^2\gamma^2 \right) \\ \Delta S_{\perp} &= \frac{V_{(d-2)}l^2b_1(d+1)}{32G_{(d+1)}b_0^2z_0^d} \left( \frac{d-1}{d+1} + \beta^2\gamma^2 \right) .\end{aligned}\tag{6.22}$$

These expressions can be used to recast Eq.(6.21) in the following form

$$\Delta C_{\parallel}^{(1)} = -\frac{\pi(d-2)}{2(d-1)^3b_0^2} \left[ \Delta S_{\parallel} - \frac{b_0^2}{(d+1)b_1^2} \Delta S_{\perp} \right] .\tag{6.23}$$

This indicates that up to first order in perturbation, the change in HSC (for a strip parallel to the direction of boost) depends upon both, the change in HEE along parallel, as well as, the perpendicular direction of boost with respect to the strip. This is an interesting feature.

## 6.2.2 Strip perpendicular to the direction of boost

In this subsection, we shall consider the case where the strip is perpendicular to the direction of boost. We shall follow the same procedure of the previous subsection to compute the HSC. In this case, the strip is considered to be along  $x^1$  - direction with  $-l/2 \leq x^1 \leq l/2$ . We parametrize the hypersurface in bulk as  $x^1 = x^1(z)$ . The area of the hypersurface in the bulk is given by

$$A_{\perp} = 2V_{(d-2)} \int_{\delta}^{z_*^{\perp}} \frac{dz}{z^{d-1}} \sqrt{K(z)} \sqrt{\frac{1}{f(z)} + (\partial_z x^1)^2}\tag{6.24}$$

where  $\delta$  is the UV cutoff and  $z_*^{\perp}$  is the turning point of the hypersurface inside the bulk. The profile of the hypersurface is given by

$$\frac{dx^1}{dz} = \left( \frac{z}{z_*^{\perp}} \right)^{d-1} \frac{1}{\sqrt{f(z)} \sqrt{\frac{K(z)}{K_*} - \left( \frac{z}{z_*^{\perp}} \right)^{2d-2}}},\tag{6.25}$$

which have been obtained by the standard process of extremization. With the identification  $x^1(0) = \frac{l}{2}$ , we get

$$\frac{l}{2} = \int_0^{z_*^\perp} dz \left( \frac{z}{z_*^\perp} \right)^{d-1} \frac{1}{\sqrt{f(z)} \sqrt{\frac{K(z)}{K_*} - \left( \frac{z}{z_*^\perp} \right)^{2d-2}}} . \quad (6.26)$$

With the assumption of finite boost and thin strip, we may use the following approximations

$$\left( \frac{z_*^\perp}{z_0} \right)^d \ll 1 \quad , \quad \beta^2 \gamma^2 \left( \frac{z_*^\perp}{z_0} \right)^d \ll 1 \quad , \quad (6.27)$$

to expand the integral in Eq.(6.26) around the pure AdS in the following manner

$$\begin{aligned} \frac{l}{2} &= z_*^\perp \int_0^1 dt t^{d-1} \frac{1}{\sqrt{R}} \left[ 1 + \frac{1}{2} x^d t^d + \frac{1}{2} y^d \frac{1-t^d}{R} + \dots \right] \\ &= z_*^\perp \left( b_0 + \frac{1}{2} (x^d b_1 + y^d I_l) \right) + \dots \end{aligned} \quad (6.28)$$

where,  $t = \frac{z}{z_*^\perp}$ ,  $R \equiv 1 - t^{2d-2}$ ,  $x = \left( \frac{z_*^\perp}{z_0} \right)$ ,  $y^d = \beta^2 \gamma^2 \left( \frac{z_*^\perp}{z_0} \right)^d$ , and the dots indicate higher order terms in  $\left( \frac{z_*^\perp}{z_0} \right)^d$ .

In order to express the new turning point  $z_*^\perp$  in terms of turning point  $z_*^{(0)}$  of pure *AdS* spacetime, we use Eqs.(6.28) and (6.4) to get

$$z_*^\perp = \frac{l/2}{b_0 + \frac{1}{2} (x^d b_1 + y^d I_l)} \simeq \frac{z_*^{(0)}}{1 + \frac{1}{2} (\bar{x}^d \frac{b_1}{b_0} + \frac{\bar{y}^d}{b_0} I_l)} , \quad (6.29)$$

where  $\bar{y}^d = \beta^2 \gamma^2 \left( \frac{z_*^{(0)}}{z_0} \right)^d$  and  $\bar{x} = \frac{z_*^{(0)}}{z_0}$ . Note that we have kept only linear order terms due to the thin strip approximation. Also note that the the turning points  $z_*^\perp$  and  $z_*^\parallel$  assumes the same value in  $\beta = 0$  limit.

Now the volume enclosed by the extremal RT-surface is given by

$$\begin{aligned} V_\perp &= 2V_{(d-2)} \int_\delta^{z_*^\perp} \frac{dz}{z^d} \sqrt{\frac{K(z)}{f(z)}} \int_z^{z_*^\perp} dz \left( \frac{u}{z_*^\perp} \right)^{d-1} \frac{1}{\sqrt{f(u)} \sqrt{\frac{K(u)}{K_*} - \left( \frac{u}{z_*^\perp} \right)^{2d-2}}} \\ &= \frac{2V_{(d-2)}}{z_*^{\perp d-2}} \int_{\frac{\delta}{z_*^\perp}}^1 \frac{dt}{t^d} \sqrt{\frac{K(t)}{f(t)}} \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)} \sqrt{\frac{K(w)}{K_*} - w^{2d-2}}} . \end{aligned} \quad (6.30)$$

Again we can use the approximation of Eq.(6.27) to expand the functions  $(K, f)$  to write the

volume as

$$\begin{aligned}
V_{\perp} &= \frac{2V_{(d-2)}}{z_*^{\perp d-2}} \int_{\frac{\delta}{z_*^{\perp}}}^1 \frac{dt}{t^d} \left( 1 + \frac{t^d}{2}(x^d + y^d) \right) \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)} \sqrt{\frac{K(w)}{K_*} - w^{2d-2}}} \\
&= \frac{2V_{(d-2)}}{z_*^{\perp d-2}} \int_{\frac{\delta}{z_*^{\perp}}}^1 \frac{dt}{t^d} \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)} \sqrt{\frac{K(w)}{K_*} - w^{2d-2}}} \\
&\quad + \frac{V_{(d-2)}}{z_*^{\perp d-2}} (x^d + y^d) \int_{\frac{\delta}{z_*^{\perp}}}^1 dt \int_t^1 dw w^{d-1} \frac{1}{\sqrt{f(w)} \sqrt{\frac{K(w)}{K_*} - w^{2d-2}}}. \tag{6.31}
\end{aligned}$$

After evaluating above integrals we use Eq.(6.29) to express  $V_{\perp}$  in terms of the volume  $V_0$  of the pure  $AdS$  spacetime. This leads to

$$\begin{aligned}
V_{\perp} &= V_{(0)} - \frac{V_{(d-2)} \bar{x}^d}{(d-1)z_*^{(0)d-2}} \left( \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + c_0 \right) \\
&\quad - \frac{V_{(d-2)} \bar{y}^d}{(d-1)z_*^{(0)d-2}} \left( \frac{(d-2)\pi I_l}{2(d-1)b_0^2} + c_2 \right) + \frac{V_{(d-2)} c_0}{z_*^{(0)d-2}} (\bar{x}^d + \bar{y}^d), \tag{6.32}
\end{aligned}$$

where only first order terms in  $\bar{x}^d$  and  $\bar{y}^d$  are kept. Hence, the change in volume in terms of the strip length  $l$  is given by

$$\begin{aligned}
\Delta V_{\perp} &\equiv V_{\perp} - V_{(0)} \\
&= -\frac{V_{(d-2)} l^2}{4b_0^2 (d-1) z_0^d} \left[ \left( \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2-d)c_0 \right) \right. \\
&\quad \left. + \beta^2 \gamma^2 \left( \frac{(d-2)\pi I_l}{2(d-1)b_0^2} + c_2 - (d-1)c_0 \right) \right]. \tag{6.33}
\end{aligned}$$

Therefore, the change in HSC is as follows

$$\begin{aligned}
\Delta C_{\perp}^{(1)} &= \frac{\Delta V_{\perp}}{8\pi G_{(d+1)}} \\
&= -\frac{V_{(d-2)} l^2}{32\pi G_{(d+1)} b_0^2 (d-1) z_0^d} \\
&\quad \times \left[ \left( \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2-d)c_0 \right) + \beta^2 \gamma^2 \left( \frac{(d-2)\pi I_l}{2(d-1)b_0^2} + c_2 - (d-1)c_0 \right) \right], \tag{6.34}
\end{aligned}$$

which can be recast in terms of  $\Delta C_{\parallel}^{(1)}$  in the following way

$$\Delta C_{\perp}^{(1)} = \Delta C_{\parallel}^{(1)} - \frac{V_{(d-2)} l^2 \beta^2 \gamma^2 c_0}{32\pi G_{(d+1)} (d-1) b_0^2 z_0^d} \left[ 1 + (d-2)(d+1) \frac{b_1^2}{b_0^2} \right]. \tag{6.35}$$

This shows that the asymmetry in HSC for a boosted black brane arises due to the presence of the boost parameter  $\beta$ . In the limit  $\beta = 0$ , the change in HSC is same for the strip being parallel or perpendicular to the direction of boost. Further Eq.(6.34) can be written in terms of  $\Delta S_{\parallel}$  and  $\Delta S_{\perp}$  as

$$\Delta C_{\perp}^{(1)} = \frac{1}{2(d-1)^3} \left[ \frac{\Delta S_{\parallel}}{(d+1)b_1^2} - \Delta S_{\perp} \left( \frac{d-2}{b_0^2} - \frac{d-3}{(d+1)b_1^2} \right) \right]. \quad (6.36)$$

This confirms that  $\Delta C_{\perp}^{(1)}$  depends upon both, the change in HEE along perpendicular, as well as, the parallel direction of boost with respect to the strip.

We have already observed the asymmetry in HSC for the boosted black brane geometry. In order to get further insight in this asymmetry, we define a quantity

$$\mathcal{R}_C = \frac{\Delta C_{\perp}^{(1)} - \Delta C_{\parallel}^{(1)}}{\Delta C_{\perp}^{(1)} + \Delta C_{\parallel}^{(1)}}. \quad (6.37)$$

Using Eqs.(6.23) and (6.36), we recast Eq.(6.37) in the following way

$$\mathcal{R}_C = \frac{\left[ \frac{2-d}{2(d-1)^3 b_0^2} - \frac{1}{2(d-1)^3 (d+1) b_1^2} \right] \mathcal{A}}{\frac{2-d}{2(d-1)^3 b_0^2} + \frac{\mathcal{R}+2d-5}{2(d-1)^3 (d+1) b_1^2}}, \quad (6.38)$$

where  $\mathcal{R}$  and  $\mathcal{A}$  are given by [81],

$$\mathcal{R} = \frac{\Delta S_{\parallel}}{\Delta S_{\perp}} = \frac{1 + \frac{2}{d-1} \beta^2 \gamma^2}{1 + \frac{d+1}{d-1} \beta^2 \gamma^2}, \quad (6.39)$$

$$\begin{aligned} \mathcal{A} &= \frac{\Delta S_{\perp} - \Delta S_{\parallel}}{\Delta S_{\perp} + \Delta S_{\parallel}} = \frac{1 - \mathcal{R}}{1 + \mathcal{R}} \\ &= \frac{\beta^2 \gamma^2}{2 + \frac{d+3}{d-1} \beta^2 \gamma^2}. \end{aligned} \quad (6.40)$$

One can easily verify that  $\mathcal{R}_C \geq 0$  for  $\beta \geq 0$ . This implies that the non-zero boost parameter is responsible for the difference in HSC for parallel and perpendicular directions of boost with respect to the strip. Further, Eq.(6.38) suggests that the asymmetry in HSC is related to the asymmetry in HEE. The asymmetry in HEE arises due to the different values of entanglement pressure in parallel and perpendicular directions [81]. It has been also observed in [91] that the origin of asymmetry in HEE for spatially anisotropic field theory depends upon the difference

in entanglement pressure in different directions. Moreover, the results obtained using the complexity equals action proposal also supports this point of view [92].

Let us now look into the possible maximum and minimum values of  $\mathcal{R}_C$ . Eq.(6.38) suggests that the maximum values of  $\mathcal{A}$  and  $\mathcal{R}$  determine the maximum value of  $\mathcal{R}_C$ . Note that the maximum values of  $\mathcal{A}$  and  $\mathcal{R}$  are obtained in the simultaneous limit  $\beta \rightarrow 1$  and  $z_0 \rightarrow \infty$ , keeping  $\frac{\beta^2 \gamma^2}{z_0^d} = \frac{1}{z_I^d} = \text{constant}$ , such that the perturbative approach remains valid. In this simultaneous limit, the boosted black brane spacetime becomes the *AdS pp*-wave spacetime, given by

$$ds^2 = \frac{L^2}{z^2} \left( -K^{-1} dt^2 + K(dy - (1 - K^{-1})dt)^2 + dx_1^2 + \dots + dx_{d-2}^2 + dz^2 \right), \quad (6.41)$$

with

$$K(z) = 1 + \frac{z^d}{z_I^d}. \quad (6.42)$$

For this geometry, the entanglement pressure is zero in all directions except the direction of wave propagation. This implies that the entanglement pressure asymmetry is maximum in this spacetime. In this spacetime  $\mathcal{R} = \frac{2}{d+1}$  and  $\mathcal{A} = \frac{d-1}{d+3}$ , which are maximum. Further, the minimum value of  $\mathcal{R}_C$  is zero when  $\beta = 0$ .

### 6.2.3 Holographic subregion complexity upto second order in perturbation

In this section, we compute the HSC in the boosted black brane geometry for a strip-like subsystem. Again we have considered the strip to be thin enough so that the hypersurface in the bulk has only penetrated the UV geometry. We want to compute the HSC upto second order in perturbation parameters  $(\frac{z_*}{z_0})^d$  and  $\beta^2 \gamma^2 (\frac{z_*}{z_0})^d$  about the pure *AdS* spacetime.

It is important to note that for a stationary spacetime one should use the covariant HEE proposal [93] instead of the RT proposal. Though the boosted black brane is a stationary spacetime, we have not used the covariant HEE proposal in the previous subsections. At the first order of perturbative expansion, the sole contribution comes from the metric perturbation [94]-[96]. Therefore, one may consider the constant time slicing in computing the HEE at the first order of perturbative expansion. However, at the second order, one can not use the

constant time slicing ( $t = \text{constant}$ ) as the deviations of the minimal surface also contribute in addition to that of the metric perturbation. But, we can still work with the RT proposal in the second order of the perturbation if we choose the strip to be perpendicular to the direction of boost [97]. This happens as the minimal surface still remain in the same time slice.

As the volume depends upon the turning point in the bulk and the turning point itself changes in the second order of perturbation, we start by computing the subsystem length  $l$ . Considering the strip to be perpendicular to the direction of boost, the length of the strip under the approximation (6.27) is the following

$$\begin{aligned}
\frac{l}{2} &= \int_0^{z_*} \frac{dz}{\sqrt{1 - \left(\frac{z}{z_0}\right)^d}} \frac{(z/z_*)^{d-1}}{\sqrt{\frac{K(z)}{K_*} - (z/z_*)^{2(d-1)}}} \\
&= z_* \left[ \int_0^1 dt \frac{t^{d-1}}{\sqrt{R}} + \frac{x^d}{2} \int_0^1 dt \frac{t^{d-1}}{\sqrt{R}} \left( t^d + \beta^2 \gamma^2 \frac{1-t^d}{R} \right) \right. \\
&\quad \left. + x^{2d} \int_0^1 dt \frac{t^{d-1}}{\sqrt{R}} \left( \frac{3}{8} t^{2d} + \frac{\beta^2 \gamma^2 t^d (1-t^d)}{4R} + \beta^4 \gamma^4 \left( \frac{3}{8} \frac{(1-t^d)^2}{R^2} - \frac{1}{2} \frac{1-t^d}{R} \right) \right) \right] \\
&= z_* \left[ b_0 + \frac{x^d}{2} (b_1 + \beta^2 \gamma^2 I_l) + x^{2d} \left( \frac{3}{8} b_2 + J_l \right) \right], \tag{6.43}
\end{aligned}$$

where  $t = \frac{z}{z_*}$ ,  $R \equiv 1 - t^{2d-2}$ ,  $x = \left(\frac{z_*}{z_0}\right)$ ,  $y^d = \beta^2 \gamma^2 \left(\frac{z_*}{z_0}\right)^d$ . The values of  $b_0$ ,  $b_1$ ,  $I_l$  and  $J_l$  are given in the Appendix. Using the Eq.(6.4), the new turning point  $z_*$  can be expressed in terms of the turning point  $z_*^{(0)}$  of the pure  $AdS$  spacetime as

$$z_* = \frac{z_*^{(0)}}{1 + \frac{1}{2b_0} (b_1 + \beta^2 \gamma^2 I_l) \bar{x}^d + \left( \frac{3b_2}{8b_0} + \frac{J_l}{b_0} - d \left( \frac{b_1 + \beta^2 \gamma^2 I_l}{2b_0} \right)^2 \right) \bar{x}^{2d}}, \tag{6.44}$$

where  $\bar{x} = z_*^{(0)}/z_0$ .

Now the expression for volume upto second order in perturbation is given by

$$\begin{aligned}
V &\simeq \frac{2V_{(d-2)}}{z_*^{d-2}} \int_{\frac{\delta}{z_*}}^1 \frac{dt}{t^d} \left( 1 + \frac{x^d + y^d}{2} t^d + \left( \frac{3}{8} x^{2d} + \frac{x^d y^d}{4} - \frac{y^{2d}}{8} \right) t^{2d} \right) \\
&\quad \times \int_t^1 dw \frac{1}{\sqrt{f(w)}} \frac{w^{d-1}}{\sqrt{\frac{K(w)}{K_*} - w^{2(d-1)}}}. \tag{6.45}
\end{aligned}$$

After computing the integrals of the above equation, we arrive at

$$\begin{aligned}
V = & V_{(0)} - \frac{V_{(d-2)}\bar{x}^d}{(d-1)\bar{z}_*^{d-2}} \left( \frac{d-2}{d-1} \frac{\pi b_1}{2b_0^2} + (2-d)c_0 \right) - \frac{V_{(d-2)}\bar{y}^d}{(d-1)\bar{z}_*^{d-2}} \left( \frac{d-2}{d-1} \frac{\pi I_l}{2b_0^2} + c_2 - (d-1)c_0 \right) \\
& - \frac{V_{(d-2)}\bar{x}^{2d}}{\bar{z}_*^{d-2}} v_{00} - \frac{V_{(d-2)}\bar{x}^d\bar{y}^d}{\bar{z}_*^{d-2}} v_{01} + \frac{V_{(d-2)}\bar{y}^{2d}}{\bar{z}_*^{d-2}} v_{11} , \tag{6.46}
\end{aligned}$$

where  $V_{(0)}$  denotes the volume for pure  $AdS$  spacetime (see Eq.(6.6)) with

$$\begin{aligned}
v_{00} &= \left( \frac{3\pi b_2}{8b_0^2} \frac{d-2}{(d-1)^2} - \frac{\pi b_1^2}{8b_0^3} \frac{(d-2)(d+3)}{(d-1)^2} + \frac{c_0 b_1}{b_0} \frac{d-2}{d-1} - \frac{c_1}{2} \frac{d^2-4}{d^2-1} \right) \\
v_{01} &= \left( \frac{b_1}{b_0} \left( c_0 - \frac{c_2}{d-1} \right) - \left( \frac{c_3}{2} + \frac{(d+2)c_1}{2(d+1)} \right) + \frac{d-2}{d-1} \frac{c_0 I_l}{b_0} + \frac{2K_1}{d-1} \right. \\
& \quad \left. + \frac{d-2}{(d-1)^2} \frac{\pi J_1}{b_0^2} - \frac{(d-2)(d+3)}{(d-1)^2} \frac{\pi b_1 I_l}{4b_0^3} \right) \\
v_{11} &= \left( \frac{c_3}{2} - \frac{c_1}{4(d+1)} + \frac{2K_2}{d-1} - \left( c_0 - \frac{c_2}{d-1} \right) \frac{I_l}{b_0} - \frac{d-2}{(d-1)^2} \frac{\pi J_2}{b_0^2} + \frac{(d-2)(d+3)}{(d-1)^2} \frac{\pi I_l^2}{8b_0^3} \right) . \tag{6.47}
\end{aligned}$$

Using eq.(6.34), we may rewrite Eq.(6.46) in the following way

$$\Delta C = \Delta C_{\perp}^{(1)} + \Delta C_{\perp}^{(2)} , \tag{6.48}$$

with

$$\Delta C_{\perp}^{(2)} = -\frac{V_{(d-2)} l^{d+2}}{8\pi G_{(d+1)} z_0^{2d} (2b_0)^{d+2}} [v_{00} + \beta^2 \gamma^2 v_{01} - \beta^4 \gamma^4 v_{11}] , \tag{6.49}$$

where  $\Delta C_{\perp}^{(1)}$  is the first order change in HSC as obtained in the previous subsection and  $\Delta C_{\perp}^{(2)}$  is the second order change in HSC. This second order change in HSC will be used to find an expression for the Fisher information metric in the next section.

### 6.3 Fisher information metric and Fidelity susceptibility

In the context of quantum information theory, there exists two well known notions of distance between two quantum states. They are quantum Fisher information metric, and the fidelity susceptibility. The Fisher information metric is defined in the following way [82],

$$G_{F,\lambda\lambda} = \langle \delta\rho | \delta\rho \rangle_{\lambda\lambda}^{(\sigma)} = \frac{1}{2} tr \left( \delta\rho \frac{d}{d(\delta\lambda)} \log(\sigma + \delta\lambda\delta\rho) |_{\delta\lambda=0} \right) \tag{6.50}$$

where the density matrix  $\sigma$  has undergone a small deviation by  $\delta\rho$ . On the other hand the fidelity susceptibility is given by

$$G_{\lambda\lambda} = \partial_{\lambda}^2 F; \quad F = \text{tr} \sqrt{\sqrt{\sigma_{\lambda}} \rho_{\lambda+\delta\lambda} \sqrt{\sigma_{\lambda}}} , \quad (6.51)$$

where  $F$  is called the Fidelity;  $\sigma$  and  $\rho$  denotes the initial and final density matrices.

The holographic prescription to compute the Fisher information metric revolves around a quantity, called the relative entropy, defined as [88]

$$S_{rel}(\rho_m \parallel \rho_0) = \Delta \langle H_{\rho_0} \rangle - \Delta S , \quad (6.52)$$

where,  $\rho_0$  is the unperturbed state,  $\rho_m$  is the perturbed state with  $m$  as perturbation parameter,  $\Delta \langle H_{\rho_0} \rangle$  is the change in modular Hamiltonian, and  $\Delta S$  is the change in entanglement entropy when one changes the state from  $\rho_0$  to  $\rho_m$ . It has been shown in [88] that at first order in perturbation parameter ( $m$ ) the relative entropy vanishes and in second order in perturbation parameter the relative entropy is given by  $S_{rel} = -\Delta S^{(2)}$ . From the relative entropy one can derive the Fisher information metric as

$$G_{F,mm} = \frac{\partial^2}{\partial m^2} S_{rel}(\rho_m \parallel \rho_0) . \quad (6.53)$$

We may now proceed to holographically compute the Fisher information metric for the pure  $AdS$  black brane. The  $AdS_{(d+1)}$  black brane metric is given by

$$ds^2 = \frac{1}{z^2} \left( -f dt^2 + dx_1^2 + \dots + dx_{d-1}^2 + \frac{dz^2}{f} \right) \quad (6.54)$$

with

$$f = 1 - \frac{z^d}{z_0^d} . \quad (6.55)$$

The inverse of the lapse function can be expanded in terms of the perturbation parameter  $m = 1/z_0^d$  as

$$\frac{1}{f(z)} = \frac{1}{1 - \frac{z^d}{z_0^d}} = 1 + mz^d + m^2 z^{2d} + \dots . \quad (6.56)$$

In terms of the parameter, the change in minimal area (up to second order in perturbation parameter) of the hypersurface in the bulk for a strip- like region lying at the boundary is given

by [79]

$$A - A_0 = \left[ \frac{V_{(d-2)} a_1 l^2}{4b_0^2} \frac{d-1}{d+1} m + \frac{V_{(d-2)} a_1 h_0 l^{d+2}}{(2b_0)^{d+2}} m^2 \right], \quad (6.57)$$

where

$$h_0 = \frac{d-1}{d+1} \left( -\frac{b_1}{2b_0} + \frac{3(d+1)}{4(2d+1)} \frac{a_2}{a_1} \right). \quad (6.58)$$

Hence, the relative entropy is given by

$$S_{rel} = -\frac{1}{4G_{(d+1)}} \left[ \frac{V_{(d-2)} a_1 h_0 l^{d+2}}{(2b_0)^{d+2}} m^2 \right]. \quad (6.59)$$

Note that  $S_{rel}$  is always positive as  $h_0$  is negative for all  $d$ . Now the fisher information metric can be derived using the Eq.(6.53), which reads

$$G_{F,mm} = \frac{\partial^2}{\partial m^2} S_{rel} = -\frac{V_{(d-2)} a_1 h_0 l^{d+2}}{2G_{(d+1)} (2b_0)^{d+2}}. \quad (6.60)$$

In [82], a proposal for computing the above quantity is given. The proposal suggests to construct a finite quantity

$$\mathcal{F} = C_d (V_{(m^2)} - V_{(0)}), \quad (6.61)$$

where  $V_{(0)}$  is the volume for pure *AdS* back ground,  $V_{m^2}$  is the volume up to second order of perturbation around the pure *AdS* background, and  $C_d$  is a dimensionless constant. Note that by volume we mean the volume under the RT extremal surface. Further, the constant  $C_d$  cannot be fixed from the first principles of the gravity side. We may now use this proposal to compute the Fisher information metric for the pure black brane. For the pure *AdS* black brane the difference in volume is given by ( $\beta = 0$  in Eq.(6.46))

$$V_{(m^2)} - V_{(0)} = - \left[ \frac{V_{(d-2)} l^2}{4b_0^2 (d-1)} m v_0 + \frac{V_{(d-2)} l^{d+2}}{(2b_0)^{d+2}} m^2 v_{00} \right], \quad (6.62)$$

with

$$v_0 = \left( \frac{d-2}{d-1} \frac{\pi b_1}{2b_0^2} + (2-d)c_0 \right). \quad (6.63)$$

The holographic dual of Fisher information metric is defined as [82]

$$G_{F,mm} = \partial_m^2 \mathcal{F}; \quad \mathcal{F} = C_d (V_{m^2} - V_{(0)}), \quad (6.64)$$

with the constant  $C_d$  is to be determined by requiring that the result from above equation must agree with the result of Fisher information metric obtained from the relative entropy. For the pure *AdS* black brane, the constant  $C_d$  can be obtained using the result in (6.60) with Eqs.(6.62) and (6.64), which is given by

$$C_d = \frac{h_0 a_1}{4G_{(d+1)} v_{00}} . \quad (6.65)$$

Now the relative entropy for the boosted black brane is given by [80]

$$S_{rel} = -\frac{1}{4G_{(d+1)}} \left[ \left( \frac{l}{2b_0} \right)^{d+2} (h_0 + h_1 \beta^2 \gamma^2 + h_2 \beta^4 \gamma^4) m^2 \right] \quad (6.66)$$

with

$$\begin{aligned} h_1 &= \left( -\frac{b_1}{b_0} + \frac{a_2}{2a_1} \right) \\ h_2 &= \frac{d+1}{d-1} \left( -\frac{b_1}{2b_0} + \frac{3a_2}{4a_1(d+1)} \right) . \end{aligned} \quad (6.67)$$

It is easy to check that  $h_1$  and  $h_2$  are negative for all  $d$ . Hence,  $S_{rel}$  is positive for all  $d$ . The Fisher information metric therefore reads

$$G_{F,mm} = \frac{\partial^2}{\partial m^2} S_{rel} = -\frac{1}{2G_{(d+1)}} \left( \frac{l}{2b_0} \right)^{d+2} (h_0 + h_1 \beta^2 \gamma^2 + h_2 \beta^4 \gamma^4) . \quad (6.68)$$

On the other hand the change in volume for the boosted black brane is given by

$$\begin{aligned} V - V_{(0)} &= -\frac{V_{(d-2)} l^2}{4b_0^2 (d-1)} \left( \frac{d-2}{d-1} \frac{\pi b_1}{2b_0^2} + (2-d)c_0 \right) m \\ &\quad - \frac{V_{(d-2)} l^2 \beta^2 \gamma^2}{4b_0^2 (d-1)} \left( \frac{d-2}{d-1} \frac{\pi I_l}{2b_0^2} + c_2 - (d-1)c_0 \right) m \\ &\quad - \frac{V_{(d-2)} l^{d+2}}{(2b_0)^{d+2}} (v_{00} + \beta^2 \gamma^2 v_{01} - \beta^4 \gamma^4 v_{11}) . \end{aligned} \quad (6.69)$$

This can be used to find the holographic dual of the Fisher information metric as defined in Eq.(6.64) with the constant  $C_d$  as follows

$$C_d = \frac{a_1}{4G_{(d+1)}} \left( \frac{h_0 + h_1 \beta^2 \gamma^2 + h_2 \beta^4 \gamma^4}{v_{00} + \beta^2 \gamma^2 v_{01} - \beta^4 \gamma^4 v_{11}} \right) . \quad (6.70)$$

Note that the constant  $C_d$  in Eq.(6.70) depends on the boost parameter  $\beta$ , where as the constant  $C_d$  for the pure *AdS* black brane (see Eq.(6.65)) depends only on the dimensionality of

the spacetime and independent of any physical parameter. Moreover, the result in Eq.(6.70) matches with the result in Eq.(6.65) in  $\beta = 0$  limit.

Let us now look at another proposal to compute the fidelity susceptibility using the holographic duality principle. If one considers that the quantum states depends on a single parameter  $\lambda$ , then for pure states, the fidelity reduces to [89]

$$\langle \Psi(\lambda) | \Psi(\lambda + \delta\lambda) \rangle = 1 - G_{\lambda\lambda}(\delta\lambda)^2 + \dots \quad (6.71)$$

Therefore, the fidelity determines how close two quantum states are. The quantity  $G_{\lambda\lambda}$ , called the fidelity susceptibility, is a measure of distance between two quantum states. The holographic formula to compute the fidelity susceptibility in  $(d+1)$  - dimensional  $AdS$  spacetime reads [89]

$$G_{\lambda\lambda} = n_{d-1} \frac{Vol(\Sigma_{max})}{R^d}, \quad (6.72)$$

where  $\Sigma_{max}$  is the maximum volume in the bulk that ends at the boundary at a fixed time slice,  $R$  is the radius of curvature of  $AdS$  spacetime and  $n_{d-1}$  is a  $\mathcal{O}(1)$  constant. The above formula have been applied successfully for the mixed states also [89, 98]. Hence we can use this proposal to compute the fidelity susceptibility for the boosted black brane.

Let us start with the pure  $AdS$  black brane. Using the metric (6.54) and the formula (6.72), the fidelity susceptibility reads

$$\begin{aligned} G_{\lambda\lambda} &= n_{d-1} L^{d-1} \int_{\delta}^{z_0} dz \frac{1}{z^d \sqrt{1 - \frac{z^d}{z_0^d}}} \\ &= \frac{n_{d-1} L^{d-1}}{z_0^{d-1}} \left[ \frac{1}{d} B\left(\frac{1-d}{d}, \frac{1}{2}\right) + \frac{z_0^{d-1}}{(d-1)\delta^{d-1}} \right]. \end{aligned} \quad (6.73)$$

Note that the above result does not matches with the Fisher information metric in Eq.(6.60), obtained from relative entropy. For the boosted black brane the fidelity susceptibility is given by

$$\begin{aligned} G_{\lambda\lambda} &= n_{d-1} L^{d-1} \int_{\delta}^{z_0} dz \frac{1}{z^d} \sqrt{\frac{K(z)}{f(z)}} \\ &= n_{d-1} L^{d-1} \left[ \frac{1}{z_0^{d-1} \sqrt{1 - \beta^2}} \int_0^1 dt \frac{1}{t^d} \sqrt{\frac{1 - \beta^2(1 - t^d)}{1 - t^d}} + \frac{1}{(d-1)\delta^{d-1}} \right] \end{aligned} \quad (6.74)$$

where  $t = z/z_0$ . Further, changing the variable to  $p = 1 - t^d$ , we get

$$\begin{aligned}
G_{\lambda\lambda} &= n_{d-1} L^{d-1} \left[ \frac{1}{d\sqrt{1-\beta^2 z_0^{d-1}}} \int_0^1 \frac{dp}{\sqrt{p}} \frac{\sqrt{1-\beta^2 p}}{(1-p)^{\frac{2d-1}{d}}} + \frac{1}{(d-1)\delta^{d-1}} \right] \\
&= n_{d-1} L^{d-1} \left[ \frac{1}{d\sqrt{1-\beta^2 z_0^{d-1}}} B\left(\frac{1}{2}, \frac{1-d}{d}\right) {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{2-d}{2d}; \beta^2\right) + \frac{1}{(d-1)\delta^{d-1}} \right].
\end{aligned} \tag{6.75}$$

This expression for fidelity susceptibility matches with the fidelity susceptibility of pure black brane in  $\beta = 0$  limit. Again note that the result in Eq.(6.75) does not match with Fisher information metric in Eq.(6.68). However, these two quantities are related in quantum information theory. The possible reason may be the difference in holographic definition of these two quantities. The proposal of fidelity susceptibility [89] involves the integration up to the horizon radius of the black brane, where as the proposal in [88] says to compute the integration up to the turning point of the hypersurface in the bulk region. Therefore, the computation of fidelity susceptibility using proposal [89] is exact, as it involves the integration up to the horizon radius. Hence, it contains the information of the full spacetime geometry of the bulk region. On the other hand, the proposal in [88] involves computation up to the second order in perturbation parameter around the pure *AdS* spacetime. This means that the result does not contain information of the full spacetime geometry, but the asymptotic region only.

## 6.4 Summary

In this chapter we have computed the HSC in a boosted black brane background for a thin strip-like entangling region. The thin strip approximation assures that the hypersurface only penetrates the UV geometry of the bulk region. This allowed us to compute the HSC up to first and second order in perturbation parameter around the pure *AdS* spacetime. In the first order of perturbation, we see an asymmetry in HSC for a strip parallel and perpendicular to the direction of boost. We see that there is a relation between the asymmetry in the HSC and the asymmetry in HEE. It is observed that the different values of entanglement pressure in different directions is responsible for the asymmetry in HSC. In the second order of perturbation, HSC

has been computed only when the strip is perpendicular to the direction of boost. After that we have computed the Fisher information metric, using the holographic proposal of relative entropy, for the boosted black brane and the pure  $AdS$  black brane. Then the results of second order change in volume have been used to compute the Fisher information metric up to an undetermined constant using the proposal [82]. The constant is determined by equating this result with the Fisher information metric obtained from relative entropy. Next, another proposal [89] is used to compute the fidelity susceptibility. It is observed that the results of the Fisher information metric do not match with the results of the fidelity susceptibility [89], though in quantum information theory they are related.

## 6.5 Appendix A: List of Beta function Identities

In this appendix we give some useful Beta function integrals which we have used in the paper.

$$\begin{aligned}
b_0 &= \int_0^1 dt t^{d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{d}{2d-2}, \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma\left(\frac{d}{2d-2}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)} \\
b_1 &= \int_0^1 dt t^{2d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{d}{d-1}, \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma\left(\frac{d}{d-1}\right)}{(d+1) \Gamma\left(\frac{1}{2} + \frac{1}{d-1}\right)} \\
b_2 &= \int_0^1 dt t^{3d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{3d}{2d-2}, \frac{1}{2}\right) \\
I_l &= \int_0^1 dt t^{d-1} (1-t^d) \frac{1}{R^{\frac{3}{2}}} = \frac{d+1}{d-1} b_1 - \frac{1}{d-1} b_0 \\
c_0 &= \int_0^1 dt \frac{t^d}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{d+1}{2(d-1)}, \frac{1}{2}\right) = \frac{\pi}{2(d^2-1)b_1} \\
c_1 &= \int_0^1 dt \frac{t^{2d}}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{2d+1}{2(d-1)}, \frac{1}{2}\right) = \frac{\pi}{2(2d+1)(d-1)b_2} \\
c_2 &= \int_0^1 dt \frac{(1-t^d)}{R^{\frac{3}{2}}} = \frac{2}{d-1} c_0 + \frac{d-2}{2(d-1)^2} B\left(\frac{1}{2(d-1)}, \frac{1}{2}\right) = \frac{\pi}{(d+1)(d-1)^2 b_1} + \frac{\pi(d-2)}{2(d-1)^2 b_0} \\
c_3 &= \int_0^1 dt \frac{t^d(1-t^d)}{R^{\frac{3}{2}}} = \frac{2}{d-1} c_0 + \frac{d+2}{d-1} c_1 \\
&\int_0^1 \frac{dt}{\sqrt{1-t^{2(d-1)}}} = \frac{\pi}{2(d-1)b_0} \\
J_l &= \int_0^1 dt t^{d-1} \left( \frac{\beta^2 \gamma^2}{4} t^d + \beta^4 \gamma^4 \left( \frac{3(1-t^d)}{8(1-t^{2(d-1)})} - \frac{1}{2} \right) \right) \frac{(1-t^d)}{R^{\frac{3}{2}}} = \beta^2 \gamma^2 J_1 + \beta^4 \gamma^4 J_2 \\
K_l &= \int_0^1 dt \left( \frac{\beta^2 \gamma^2}{4} t^d + \beta^4 \gamma^4 \left( \frac{3(1-t^d)}{8(1-t^{2(d-1)})} - \frac{1}{2} \right) \right) \frac{(1-t^d)}{R^{\frac{3}{2}}} = \beta^2 \gamma^2 K_1 - \beta^4 \gamma^4 K_2 \quad (6.76)
\end{aligned}$$

with

$$\begin{aligned}
J_1 &= \frac{1}{4(d-1)} ((2d+1)b_2 - (d+1)b_1) \\
J_2 &= \frac{1}{8(d-1)^2} ((3-2d)b_0 - 2(d+1)(3-d)b_1 + 3(2d+1)b_2) - \frac{I_l}{2} \\
K_1 &= \frac{1}{2(d-1)} c_0 + \frac{d+2}{4(d-1)} c_1 \\
K_2 &= -\frac{d-4}{4(d-1)^2} c_0 + \frac{(d+2)(d-4)}{8(d-1)^2} c_1 + \frac{d}{8(d-1)} c_2 \quad (6.77)
\end{aligned}$$

where  $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$  are the Beta-functions and we have used the identity  $B(x, \frac{1}{2})B(x + \frac{1}{2}, \frac{1}{2}) = \frac{\pi}{x}$ . Further integrals are

$$\begin{aligned}
a_0 &= \int_0^1 dt t^{-d+1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{1-d/2}{d-1}, \frac{1}{2}\right) \\
a_1 &= \int_0^1 dt t^{-d+1} \frac{t^d}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{1}{d-1}, \frac{1}{2}\right) \\
a_2 &= \int_0^1 dt t^{-d+1} \frac{t^{2d}}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{1+d/2}{d-1}, \frac{1}{2}\right) \\
I_a &= \int_0^1 dt t^{d-1} (1-t^{2d}) \frac{1}{R^{3/2}} = \frac{2d+1}{d-1} b_2 - \frac{1}{d-1} b_0 .
\end{aligned} \tag{6.78}$$

Some identities we have used are

$$b_0 = (2-d)a_0, \quad b_1 = \frac{2}{d+1}a_1, \quad b_2 = \frac{2+d}{2d+1}a_2 . \tag{6.79}$$

# Chapter 7

## Conclusions

The main goal of this thesis is to study, some of the aspects of holographic entanglement entropy (HEE) and complexity. The present thesis follows the Ryu-Takayanagi (RT) proposal to compute the entanglement entropy holographically. On the other hand, we have confined our analysis to holographic subregion complexity (HSC) only, though there exist different notions of computational complexity. Our choice is justified by the fact that HSC is proportional to the volume under the RT extremal surface.

We have started with a brief review of the modified first law of entanglement thermodynamics for the excited state of a  $(3 + 1)$  - dimensional Lifshitz spacetime. Computation of the change in HEE for a strip-like subsystem, due to the change in background geometry from the pure Lifshitz spacetime to the asymptotically Lifshitz spacetime has been shown. Then we have shown the relation between the change in HEE with the change in energy, entanglement pressure, and entanglement chemical potential. This relation is the modified law of entanglement thermodynamics.

One of our aim in this thesis is to obtain a first law like relation for the change in HSC. For that, we chose the asymptotically Lifshitz spacetime and pure Lifshitz spacetime as the bulk dual of the excited state and the ground state respectively. We observe that, up to the first order in perturbation parameter, the change in HSC is related to the change in energy and entanglement chemical potential. But, the change in HSC is found to be independent of the

change in entanglement pressure. However, the change in HEE depends on the change in energy, entanglement pressure, and entanglement chemical potential. This relation, connecting the change in HSC to the change in energy and entanglement chemical potential is analogous to the first law of entanglement thermodynamics. We further note though we have considered a strip-like subsystem, the first law like relation for the change in HSC is valid even for a disk-like subsystem. This is obvious, as the change in the shape of the subsystem will only change the coefficients of various terms in the first law like relation. Moreover, we observe that the change in HSC is related to the change in HEE. This is an important observation.

Next, we computed the HEE for a strip-like subsystem in  $(3 + 1)$  - dimensional Lifshitz black hole. We observe that the finite part of the HEE in the ultraviolet limit depended on the subsystem length  $l$  as  $S_{finite}^{(UV)} \sim \frac{1}{l} (constant + \mathcal{O}(l^2))$ . This is in contrast with the result corresponding to that of a  $(3 + 1)$  - dimensional Swartzchild  $AdS$  black hole, for which  $S_{finite}^{(UV)} \sim \frac{1}{l} (constant + \mathcal{O}(l^3))$ . We further observe that the leading contribution to the HEE in the infrared limit comes from the black hole entropy. It motivates us to introduce the notion of a generalized temperature. The generalized temperature ( $T_g$ ) is being defined in terms of the renormalized entanglement entropy ( $S_{REE}$ ) as,  $\frac{1}{T_g} = \frac{S_{REE}}{2E}$ , where  $E$  is the internal energy. This has been done in view of the first law of black hole thermodynamics ( $E = \frac{T_h S_h}{2}$ ). We see that, in the infrared limit, the generalized temperature is nothing but the thermal temperature (black hole temperature). Moreover, we find that the generalized temperature is non-zero when the subsystem size becomes zero ( $\frac{l}{r_h} = 0$ , but  $r_h \neq 0$ ). This is completely a non-relativistic phenomenon.

Holographic entanglement entropy for a strip-like subsystem in the context of a charged black hole (Reissner-Nordstrom black hole in  $AdS$  spacetime) in arbitrary dimension has also been computed. We observe that there exists a lower bound on the horizon radius due to the extremality condition of the Hawking temperature. In a canonical ensemble with a fixed black hole charge ( $Q$ ), we have obtained the expressions for the HEE in different possible regimes. We obtained a first law of entanglement thermodynamics considering the extremal  $AdS$ -Reissner Nordstrom black hole with small charge as the bulk dual of a ground state and the non-extremal

black hole (in low temperature and small charge limit) as the gravity dual of the excited state. We observe that the entanglement temperature is dependent on the dimension of the spacetime geometry.

We have then looked at the HSC for a boosted black brane with a strip-like subsystem. As the boost can be in a direction parallel or perpendicular to the direction of the strip length, we have analyzed both of the cases. We observe that there exists an asymmetry in the subregion complexities up to first order in perturbation parameter. This asymmetry in holographic subregion complexity arises due to the asymmetry in holographic entanglement entropies in the parallel and perpendicular directions. We further observe that the asymmetry in holographic entanglement entropy owes its origin to the different values of entanglement pressure in parallel and perpendicular directions. Hence, the unequal entanglement pressure is the cause of asymmetry in holographic subregion complexity. After that, we have holographically obtained the expressions for the Fisher information metric and the fidelity susceptibility for the pure black brane as well as for the boosted black brane. We observe that the expressions for the fidelity susceptibility do not match with the expressions for the Fisher information metric. This is in sharp contrast with the results present in the quantum information literature. In quantum information literature, the Fisher information metric and the fidelity susceptibility are related to each other in general. This is an important observation of the present thesis.

In the future, we hope to extend our analysis connecting the change in HSC to the change in energy and entanglement chemical potential for general perturbations to the pure Lifshitz spacetime. It would be very interesting to extend our analysis for Lifshitz black holes and hyperscale violating Lifshitz theories.

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